

Original Article

New Numerical Algorithm for Integrations of Volterra Integral Equations of Second Kind

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Abstract - This study presented a New Numerical Scheme (NNS) using a linear block algorithm for solving Volterra integro-differential equations of the second kind. The NNS was derived using a linear block algorithm and analyzed with respect to key numerical properties, including order and error constant, consistency, zero-stability, convergence, and region of absolute stability. Theoretical analysis confirmed that the new numerical scheme is of uniformly orders, consistent, zero-stable, and possesses an A-stability region. To validate its performance, the NNS is applied to both linear and nonlinear Volterra integro-differential equations of the second kind and compared with existing methods such as the Adams-Bashforth-Moulton method, trapezoidal-based approaches, and block methods. The numerical results reveal that the NNS consistently produced solutions close to the exact values, outperforming other established methods across various step sizes. These findings demonstrate that the NNS is not only mathematically reliable but also computationally efficient, making it a robust tool for solving a wide range of problems involving Volterra integro-differential equations of the second kind.

Keywords - Volterra Integro-Differential Equations, New Numerical Scheme, Convergence, Numerical integration, Linear and nonlinear problems.

1. Introduction

In engineering and the sciences, mathematical models often employ differential and integral equations to describe natural and physical processes, with equations capturing diverse system dynamics and integral equations addressing processes influenced by past states. Depending on their limits of integration, integral equations are classified as Fredholm or Volterra and are widely applied in fields such as fluid flow, heat conduction, and electromagnetism [1]. Due to their mathematical complexity, several analytical and numerical methods, including the Adomian Decomposition Method and collocation techniques, have been developed to obtain accurate and efficient solutions [1, 2]. Volterra Integro-Differential Equations (VIDEs), which combine features of both differential and integral equations, model systems where future behavior depends not only on present changes but also on accumulated past states [3]. This structure makes VIDEs especially useful in representing processes with memory effects, such as population dynamics, viscoelastic materials, and heat transfer [4, 5].

Essentially, they capture the historical dependence of a system's behavior and are often regarded as an alternative representation of differential equations enriched by integral components [1]. The general expression for a Volterra integro-differential equation can be written as:

$$y^{(n)}(x) = f(x) - \lambda \int_0^x K(x,t)y(t)dt \quad (1)$$



With $n \geq 1$ is the initial conditions $y(0) = x_0, y_1'(0) = x_1, y''(0) = x_2, \dots, y^{n-1}(0) = x_{n-1}$.

Where $K(x, t)$ is the kernel or nucleus of the integral equation 0, and x are the limits of integration.

Classical approaches to Volterra Integro-Differential Equations (VIDEs), such as series expansion, Picard iteration, and the Laplace transform, provided the foundation for analytical solutions, particularly in linear cases [5]. With advancements, semi-analytical methods including the Adomian Decomposition Method (ADM), Homotopy Analysis Method (HAM), and Variational Iteration Method (VIM) were introduced to enhance flexibility and convergence for nonlinear problems [8-10]. Other techniques, like perturbation methods, Green's function formulations, and resolvent kernel approaches, have also been applied, while recent studies emphasize the development of more efficient numerical and hybrid methods to improve solution accuracy and robustness [11, 12].

Volterra Integral Equations (VIEs) of the form (1) remain an important research focus due to their diverse applications in science and engineering [1]. Traditional approaches, such as direct computation [13, 14], Adomian decomposition [4, 8], variational iteration [15, 16], successive approximation and substitution methods [17, 18], are widely used but face challenges, particularly the main challenges of existing numerical methods for Volterra integral equations are their high computational demands for higher-order cases and the challenges associated with implementing complex series solutions. To address these challenges, this study developed a New Numerical Scheme (NNS) using a third derivative of the Linear Block Algorithm (LBA) for solving VIEs (1.1).

2. Materials and Methods

In this section, the numerical scheme was derived using the linear block algorithm following the method in [19] for solving the Volterra Integral Equation of the second (1). Proposition 1 was adopted with the general linear multistep method of the form

$$\sum_{j=0}^1 \alpha_j y_{n+j} = h^\alpha \sum_{j=0}^1 \beta_j f_{n+j} \tag{2}$$

2.1. Proposition 2.1

The general linear multistep method (2) exists as the only one of the numerical schemes from every one-step block hybrid method. The linear block algorithm in the form of

$$y_{n+\eta} = \sum_{j=0}^2 \frac{(th)^j}{j!} y_n^{(j)} + \sum_{j=0}^1 (\Pi_{jt} f_{n+j}), \quad t = -\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1 \tag{3}$$

and the higher derivatives of (3) of the form.

$$y_{n+\eta}^a = \sum_{j=0}^{2-t} \frac{(th)^j}{j!} y_n^{(j+a)} + \sum_{j=0}^7 (\Psi_{ja} f_{n+j}), \quad a = 1 \left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1 \right), \quad a = 2 \left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1 \right) \tag{4}$$

is considered, with $\Pi_{tj} = X^{-1}P$ and $\Psi_{tja} = X^{-1}Z$.

$$X = \begin{pmatrix} \frac{1}{\left(-\frac{1}{6}h\right)^1} & \frac{1}{\left(-\frac{1}{3}h\right)^1} & 1 & \frac{1}{\left(\frac{1}{6}h\right)^1} & \frac{1}{\left(\frac{1}{3}h\right)^1} & \frac{1}{\left(\frac{1}{2}h\right)^1} & \frac{1}{\left(\frac{2}{3}h\right)^1} & \frac{1}{\left(\frac{5}{6}h\right)^1} & 1 \\ \frac{1!}{\left(-\frac{1}{6}h\right)^2} & \frac{1!}{\left(-\frac{1}{3}h\right)^2} & (0)^2 & \frac{1!}{\left(\frac{1}{6}h\right)^2} & \frac{1!}{\left(\frac{1}{3}h\right)^2} & \frac{1!}{\left(\frac{1}{2}h\right)^2} & \frac{1!}{\left(\frac{2}{3}h\right)^2} & \frac{1!}{\left(\frac{5}{6}h\right)^2} & (h)^2 \\ \frac{1!}{\left(-\frac{1}{6}h\right)^3} & \frac{1!}{\left(-\frac{1}{3}h\right)^3} & (0)^3 & \frac{1!}{\left(\frac{1}{6}h\right)^3} & \frac{1!}{\left(\frac{1}{3}h\right)^3} & \frac{1!}{\left(\frac{1}{2}h\right)^3} & \frac{1!}{\left(\frac{2}{3}h\right)^3} & \frac{1!}{\left(\frac{5}{6}h\right)^3} & (h)^3 \\ \frac{1!}{\left(-\frac{1}{6}h\right)^4} & \frac{1!}{\left(-\frac{1}{3}h\right)^4} & (0)^4 & \frac{1!}{\left(\frac{1}{6}h\right)^4} & \frac{1!}{\left(\frac{1}{3}h\right)^4} & \frac{1!}{\left(\frac{1}{2}h\right)^4} & \frac{1!}{\left(\frac{2}{3}h\right)^4} & \frac{1!}{\left(\frac{5}{6}h\right)^4} & (h)^4 \\ \frac{1!}{\left(-\frac{1}{6}h\right)^5} & \frac{1!}{\left(-\frac{1}{3}h\right)^5} & (0)^5 & \frac{1!}{\left(\frac{1}{6}h\right)^5} & \frac{1!}{\left(\frac{1}{3}h\right)^5} & \frac{1!}{\left(\frac{1}{2}h\right)^5} & \frac{1!}{\left(\frac{2}{3}h\right)^5} & \frac{1!}{\left(\frac{5}{6}h\right)^5} & (h)^5 \\ \frac{1!}{\left(-\frac{1}{6}h\right)^6} & \frac{1!}{\left(-\frac{1}{3}h\right)^6} & (0)^6 & \frac{1!}{\left(\frac{1}{6}h\right)^6} & \frac{1!}{\left(\frac{1}{3}h\right)^6} & \frac{1!}{\left(\frac{1}{2}h\right)^6} & \frac{1!}{\left(\frac{2}{3}h\right)^6} & \frac{1!}{\left(\frac{5}{6}h\right)^6} & (h)^6 \\ \frac{1!}{\left(-\frac{1}{6}h\right)^7} & \frac{1!}{\left(-\frac{1}{3}h\right)^7} & (0)^7 & \frac{1!}{\left(\frac{1}{6}h\right)^7} & \frac{1!}{\left(\frac{1}{3}h\right)^7} & \frac{1!}{\left(\frac{1}{2}h\right)^7} & \frac{1!}{\left(\frac{2}{3}h\right)^7} & \frac{1!}{\left(\frac{5}{6}h\right)^7} & (h)^7 \\ \frac{1!}{\left(-\frac{1}{6}h\right)^8} & \frac{1!}{\left(-\frac{1}{3}h\right)^8} & (0)^8 & \frac{1!}{\left(\frac{1}{6}h\right)^8} & \frac{1!}{\left(\frac{1}{3}h\right)^8} & \frac{1!}{\left(\frac{1}{2}h\right)^8} & \frac{1!}{\left(\frac{2}{3}h\right)^8} & \frac{1!}{\left(\frac{5}{6}h\right)^8} & (h)^8 \end{pmatrix} \cdot P = \begin{pmatrix} \frac{(th)^3}{3!} \\ \frac{(th)^4}{4!} \\ \frac{(th)^5}{5!} \\ \frac{(th)^6}{6!} \\ \frac{(th)^7}{7!} \\ \frac{(th)^8}{8!} \\ \frac{(th)^9}{9!} \\ \frac{(th)^{10}}{10!} \\ \frac{(th)^{11}}{11!} \end{pmatrix} \cdot Z = \begin{pmatrix} \frac{(th)^{3-a}}{(3-a)!} \\ \frac{(th)^{4-a}}{(4-a)!} \\ \frac{(th)^{5-a}}{(5-a)!} \\ \frac{(th)^{6-a}}{(6-a)!} \\ \frac{(th)^{7-a}}{(7-a)!} \\ \frac{(th)^{8-a}}{(8-a)!} \\ \frac{(th)^{9-a}}{(9-a)!} \\ \frac{(th)^{10-a}}{(10-a)!} \\ \frac{(th)^{11-a}}{(11-a)!} \end{pmatrix}$$

Proof

Solving Equations (3) and (4) one by one to obtain the polynomial

$$y(x_n + th) = \alpha_{\frac{1}{6}} y_{n+\frac{1}{6}} + \alpha_{\frac{1}{3}} y_{n+\frac{1}{3}} + \alpha_{\frac{1}{2}} y_{n+\frac{1}{2}} + h^3 \left(\beta_{\frac{1}{6}} f_{n-\frac{1}{6}} + \beta_{\frac{1}{3}} f_{n-\frac{1}{3}} + \beta_0 f_n + \beta_{\frac{1}{6}} f_{n+\frac{1}{6}} + \beta_{\frac{1}{3}} f_{n+\frac{1}{3}} + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_{\frac{2}{3}} f_{n+\frac{2}{3}} + \beta_{\frac{5}{6}} f_{n+\frac{5}{6}} + \beta_1 f_{n+1} \right) \tag{5}$$

Where $t = x_n + xh$ in the polynomial (5) and

$$\begin{aligned}
 \alpha_{\frac{1}{6}} &= 3 - 15t + 18t^2, \alpha_{\frac{1}{3}} = 3 + 24t - 36t^2, \alpha_{\frac{1}{2}} = 1 - 9t + 18t^2, \\
 \beta_{-\frac{1}{6}} &= -\frac{827}{65318400} - \frac{42841}{718502400}t + \frac{8999}{5443200}t^2 - \frac{1}{14}t^4 + \frac{117}{350}t^5 - \frac{53}{100}t^6 - \frac{33}{170}t^7 + \frac{99}{56}t^8 - \frac{171}{70}t^9 + \frac{513}{350}t^{10} - \frac{648}{19254}t^{11} \\
 \beta_{-\frac{1}{3}} &= \frac{331}{261273600} + \frac{3551}{2874009600}t - \frac{697}{6220800}t^2 - \frac{1}{224}t^4 - \frac{87}{5600}t^5 - \frac{1}{160}t^6 + \frac{381}{2800}t^7 - \frac{9}{28}t^8 + \frac{99}{280}t^9 - \frac{27}{140}t^{10} + \frac{81}{1925}t^{11} \\
 \beta_0 &= \frac{631}{32659200} + \frac{1560457}{718502400}t - \frac{81625}{241920}t^2 + \frac{1}{6}t^3 - \frac{19}{80}t^4 - \frac{331}{600}t^5 + \frac{819}{400}t^6 - \frac{99}{100}t^7 + \frac{81}{20}t^8 - \frac{513}{70}t^9 + \frac{243}{50}t^{10} - \frac{648}{275}t^{11} \\
 \beta_{\frac{1}{6}} &= -\frac{149189}{65318400} + \frac{4051717}{143700480}t - \frac{539927}{5443200}t^2 + \frac{1}{2}t^4 + \frac{3}{50}t^5 - \frac{313}{100}t^6 + \frac{579}{175}t^7 + \frac{279}{56}t^8 - \frac{873}{70}t^9 + \frac{459}{50}t^{10} - \frac{648}{275}t^{11} \\
 \beta_{\frac{1}{3}} &= -\frac{63317}{26127360} + \frac{854819}{41057280}t - \frac{1087}{34020}t^2 - \frac{5}{16}t^4 + \frac{27}{80}t^5 + \frac{103}{40}t^6 - \frac{1221}{280}t^7 - \frac{99}{28}t^8 + \frac{369}{28}t^9 - \frac{54}{5}t^{10} + \frac{162}{55}t^{11} \\
 \beta_{\frac{1}{2}} &= \frac{7331}{65318400} - \frac{290999}{718502400}t - \frac{1157}{259200}t^2 + \frac{1}{6}t^4 - \frac{37}{150}t^5 - \frac{27}{20}t^6 + \frac{549}{175}t^7 + \frac{81}{56}t^8 - \frac{621}{70}t^9 + \frac{81}{10}t^{10} - \frac{648}{275}t^{11} \\
 \beta_{\frac{2}{3}} &= -\frac{857}{16329600} + \frac{177581}{718502400}t + \frac{31429}{21772800}t^2 - \frac{1}{16}t^4 + \frac{21}{200}t^5 + \frac{197}{400}t^6 - \frac{933}{700}t^7 - \frac{9}{28}t^8 + \frac{261}{70}t^9 - \frac{189}{50}t^{10} + \frac{324}{275}t^{11} \\
 \beta_{\frac{5}{6}} &= \frac{733}{65318400} - \frac{35977}{718502400}t - \frac{1831}{5443200}t^2 + \frac{1}{70}t^4 - \frac{9}{350}t^5 - \frac{11}{100}t^6 + \frac{57}{175}t^7 + \frac{9}{280}t^8 - \frac{9}{10}t^9 + \frac{351}{250}t^{10} - \frac{648}{1925}t^{11} \\
 \beta_1 &= -\frac{289}{261273600} - \frac{2699}{574801920}t + \frac{517}{14515200}t^2 - \frac{1}{672}t^4 + \frac{47}{16800}t^5 + \frac{9}{800}t^6 - \frac{99}{2800}t^7 + \frac{27}{280}t^9 - \frac{81}{700}t^{10} + \frac{81}{1925}t^{11}
 \end{aligned}$$

The generalized algorithm (3) is expanded to gives the new numerical scheme as

$$\left. \begin{aligned}
 y'_{n-\frac{1}{6}} &= y'_n - \frac{1}{6}hy''_n + \frac{\left(-\frac{1}{6}h\right)^2}{2!}y'''_n + h^3\left(\Pi_{10}f_{n-\frac{1}{6}} + \Pi_{11}f_{n-\frac{1}{3}} + \Pi_{12}f_n + \Pi_{13}f_{n+\frac{1}{6}} + \Pi_{14}f_{n+\frac{1}{3}} + \Pi_{15}f_{n+\frac{1}{2}} + \Pi_{16}f_{n+\frac{2}{3}} + \Pi_{17}f_{n+\frac{5}{6}} + \Pi_{18}f_{n+1}\right) \\
 y'_{n-\frac{1}{3}} &= y'_n - \frac{1}{3}hy''_n + \frac{\left(-\frac{1}{3}h\right)^2}{2!}y'''_n + h^3\left(\Pi_{20}f_{n-\frac{1}{6}} + \Pi_{21}f_{n-\frac{1}{3}} + \Pi_{22}f_n + \Pi_{23}f_{n+\frac{1}{6}} + \Pi_{24}f_{n+\frac{1}{3}} + \Pi_{25}f_{n+\frac{1}{2}} + \Pi_{26}f_{n+\frac{2}{3}} + \Pi_{27}f_{n+\frac{5}{6}} + \Pi_{28}f_{n+1}\right) \\
 y'_{n+\frac{1}{6}} &= y'_n + \frac{1}{6}hy''_n + \frac{\left(\frac{1}{6}h\right)^2}{2!}y'''_n + h^3\left(\Pi_{30}f_{n-\frac{1}{6}} + \Pi_{31}f_{n-\frac{1}{3}} + \Pi_{32}f_n + \Pi_{33}f_{n+\frac{1}{6}} + \Pi_{34}f_{n+\frac{1}{3}} + \Pi_{35}f_{n+\frac{1}{2}} + \Pi_{36}f_{n+\frac{2}{3}} + \Pi_{37}f_{n+\frac{5}{6}} + \Pi_{38}f_{n+1}\right) \\
 y'_{n+\frac{1}{3}} &= y'_n + \frac{1}{3}hy''_n + \frac{\left(\frac{1}{3}h\right)^2}{2!}y'''_n + h^3\left(\Pi_{40}f_{n-\frac{1}{6}} + \Pi_{41}f_{n-\frac{1}{3}} + \Pi_{42}f_n + \Pi_{43}f_{n+\frac{1}{6}} + \Pi_{44}f_{n+\frac{1}{3}} + \Pi_{45}f_{n+\frac{1}{2}} + \Pi_{46}f_{n+\frac{2}{3}} + \Pi_{47}f_{n+\frac{5}{6}} + \Pi_{48}f_{n+1}\right) \\
 y'_{n+\frac{1}{2}} &= y'_n + \frac{1}{2}hy''_n + \frac{\left(\frac{1}{2}h\right)^2}{2!}y'''_n + h^3\left(\Pi_{50}f_{n-\frac{1}{6}} + \Pi_{51}f_{n-\frac{1}{3}} + \Pi_{52}f_n + \Pi_{53}f_{n+\frac{1}{6}} + \Pi_{54}f_{n+\frac{1}{3}} + \Pi_{55}f_{n+\frac{1}{2}} + \Pi_{56}f_{n+\frac{2}{3}} + \Pi_{57}f_{n+\frac{5}{6}} + \Pi_{58}f_{n+1}\right) \\
 y'_{n+\frac{2}{3}} &= y'_n + \frac{2}{3}hy''_n + \frac{\left(\frac{2}{3}h\right)^2}{2!}y'''_n + h^3\left(\Pi_{60}f_{n-\frac{1}{6}} + \Pi_{61}f_{n-\frac{1}{3}} + \Pi_{62}f_n + \Pi_{63}f_{n+\frac{1}{6}} + \Pi_{64}f_{n+\frac{1}{3}} + \Pi_{65}f_{n+\frac{1}{2}} + \Pi_{66}f_{n+\frac{2}{3}} + \Pi_{67}f_{n+\frac{5}{6}} + \Pi_{68}f_{n+1}\right) \\
 y'_{n+\frac{5}{6}} &= y'_n + \frac{5}{6}hy''_n + \frac{\left(\frac{5}{6}h\right)^2}{2!}y'''_n + h^3\left(\Pi_{70}f_{n-\frac{1}{6}} + \Pi_{71}f_{n-\frac{1}{3}} + \Pi_{72}f_n + \Pi_{73}f_{n+\frac{1}{6}} + \Pi_{74}f_{n+\frac{1}{3}} + \Pi_{75}f_{n+\frac{1}{2}} + \Pi_{76}f_{n+\frac{2}{3}} + \Pi_{77}f_{n+\frac{5}{6}} + \Pi_{78}f_{n+1}\right) \\
 y'_{n+1} &= y'_n + y''_n + \frac{(h)^2}{2!}y'''_n + h^3\left(\Pi_{80}f_{n-\frac{1}{6}} + \Pi_{81}f_{n-\frac{1}{3}} + \Pi_{82}f_n + \Pi_{83}f_{n+\frac{1}{6}} + \Pi_{84}f_{n+\frac{1}{3}} + \Pi_{85}f_{n+\frac{1}{2}} + \Pi_{86}f_{n+\frac{2}{3}} + \Pi_{87}f_{n+\frac{5}{6}} + \Pi_{88}f_{n+1}\right)
 \end{aligned} \right\} \tag{6}$$

The higher derivatives of the algorithm (4) are expanded as

$$\left. \begin{aligned}
 y'_{n-\frac{1}{6}} &= y'_n - \frac{1}{6}hy''_n + h^2f\left(\Psi_{110}f_{n-\frac{1}{6}} + \Psi_{111}f_{n-\frac{1}{3}} + \Psi_{112}f_n + \Psi_{113}f_{n+\frac{1}{6}} + \Psi_{114}f_{n+\frac{1}{3}} + \Psi_{115}f_{n+\frac{1}{2}} + \Psi_{116}f_{n+\frac{2}{3}} + \Psi_{117}f_{n+\frac{5}{6}} + \Psi_{118}f_{n+1}\right) \\
 y'_{n-\frac{1}{3}} &= y'_n - \frac{1}{3}hy''_n + h^2f\left(\Psi_{120}f_{n-\frac{1}{6}} + \Psi_{121}f_{n-\frac{1}{3}} + \Psi_{122}f_n + \Psi_{123}f_{n+\frac{1}{6}} + \Psi_{124}f_{n+\frac{1}{3}} + \Psi_{125}f_{n+\frac{1}{2}} + \Psi_{126}f_{n+\frac{2}{3}} + \Psi_{127}f_{n+\frac{5}{6}} + \Psi_{128}f_{n+1}\right) \\
 y'_{n+\frac{1}{6}} &= y'_n + \frac{1}{6}hy''_n + h^2f\left(\Psi_{130}f_{n-\frac{1}{6}} + \Psi_{131}f_{n-\frac{1}{3}} + \Psi_{132}f_n + \Psi_{133}f_{n+\frac{1}{6}} + \Psi_{134}f_{n+\frac{1}{3}} + \Psi_{135}f_{n+\frac{1}{2}} + \Psi_{136}f_{n+\frac{2}{3}} + \Psi_{137}f_{n+\frac{5}{6}} + \Psi_{138}f_{n+1}\right) \\
 y'_{n+\frac{1}{3}} &= y'_n + \frac{1}{3}hy''_n + h^2f\left(\Psi_{140}f_{n-\frac{1}{6}} + \Psi_{141}f_{n-\frac{1}{3}} + \Psi_{142}f_n + \Psi_{143}f_{n+\frac{1}{6}} + \Psi_{144}f_{n+\frac{1}{3}} + \Psi_{145}f_{n+\frac{1}{2}} + \Psi_{146}f_{n+\frac{2}{3}} + \Psi_{147}f_{n+\frac{5}{6}} + \Psi_{148}f_{n+1}\right) \\
 y'_{n+\frac{1}{2}} &= y'_n + \frac{1}{2}hy''_n + h^2f\left(\Psi_{150}f_{n-\frac{1}{6}} + \Psi_{151}f_{n-\frac{1}{3}} + \Psi_{152}f_n + \Psi_{153}f_{n+\frac{1}{6}} + \Psi_{154}f_{n+\frac{1}{3}} + \Psi_{155}f_{n+\frac{1}{2}} + \Psi_{156}f_{n+\frac{2}{3}} + \Psi_{157}f_{n+\frac{5}{6}} + \Psi_{158}f_{n+1}\right) \\
 y'_{n+\frac{2}{3}} &= y'_n + \frac{2}{3}hy''_n + h^2f\left(\Psi_{160}f_{n-\frac{1}{6}} + \Psi_{161}f_{n-\frac{1}{3}} + \Psi_{162}f_n + \Psi_{163}f_{n+\frac{1}{6}} + \Psi_{164}f_{n+\frac{1}{3}} + \Psi_{165}f_{n+\frac{1}{2}} + \Psi_{166}f_{n+\frac{2}{3}} + \Psi_{167}f_{n+\frac{5}{6}} + \Psi_{168}f_{n+1}\right) \\
 y'_{n+\frac{5}{6}} &= y'_n + \frac{5}{6}hy''_n + h^2f\left(\Psi_{170}f_{n-\frac{1}{6}} + \Psi_{171}f_{n-\frac{1}{3}} + \Psi_{172}f_n + \Psi_{173}f_{n+\frac{1}{6}} + \Psi_{174}f_{n+\frac{1}{3}} + \Psi_{175}f_{n+\frac{1}{2}} + \Psi_{176}f_{n+\frac{2}{3}} + \Psi_{177}f_{n+\frac{5}{6}} + \Psi_{178}f_{n+1}\right) \\
 y'_{n+1} &= y'_n + hy''_n + h^2f\left(\Psi_{180}f_{n-\frac{1}{6}} + \Psi_{181}f_{n-\frac{1}{3}} + \Psi_{182}f_n + \Psi_{183}f_{n+\frac{1}{6}} + \Psi_{184}f_{n+\frac{1}{3}} + \Psi_{185}f_{n+\frac{1}{2}} + \Psi_{186}f_{n+\frac{2}{3}} + \Psi_{187}f_{n+\frac{5}{6}} + \Psi_{188}f_{n+1}\right)
 \end{aligned} \right\} \tag{7}$$

$$\left. \begin{aligned}
 y''_{n-\frac{1}{6}} &= y''_n + h \left(\Psi_{210}f_{n-\frac{1}{6}} + \Psi_{211}f_{n-\frac{1}{3}} + \Psi_{212}f_n + \Psi_{213}f_{n+\frac{1}{6}} + \Psi_{214}f_{n+\frac{1}{3}} + \Psi_{215}f_{n+\frac{1}{2}} + \Psi_{216}f_{n+\frac{2}{3}} + \Psi_{217}f_{n+\frac{5}{6}} + \Psi_{218}f_{n+1} \right) \\
 y''_{n-\frac{1}{3}} &= y''_n + h \left(\Psi_{220}f_{n-\frac{1}{6}} + \Psi_{221}f_{n-\frac{1}{3}} + \Psi_{222}f_n + \Psi_{223}f_{n+\frac{1}{6}} + \Psi_{224}f_{n+\frac{1}{3}} + \Psi_{225}f_{n+\frac{1}{2}} + \Psi_{226}f_{n+\frac{2}{3}} + \Psi_{227}f_{n+\frac{5}{6}} + \Psi_{228}f_{n+1} \right) \\
 y''_{n+\frac{1}{6}} &= y''_n + h \left(\Psi_{230}f_{n-\frac{1}{6}} + \Psi_{231}f_{n-\frac{1}{3}} + \Psi_{232}f_n + \Psi_{233}f_{n+\frac{1}{6}} + \Psi_{234}f_{n+\frac{1}{3}} + \Psi_{235}f_{n+\frac{1}{2}} + \Psi_{236}f_{n+\frac{2}{3}} + \Psi_{237}f_{n+\frac{5}{6}} + \Psi_{238}f_{n+1} \right) \\
 y''_{n+\frac{1}{3}} &= y''_n + h \left(\Psi_{240}f_{n-\frac{1}{6}} + \Psi_{241}f_{n-\frac{1}{3}} + \Psi_{242}f_n + \Psi_{243}f_{n+\frac{1}{6}} + \Psi_{244}f_{n+\frac{1}{3}} + \Psi_{245}f_{n+\frac{1}{2}} + \Psi_{246}f_{n+\frac{2}{3}} + \Psi_{247}f_{n+\frac{5}{6}} + \Psi_{248}f_{n+1} \right) \\
 y''_{n+\frac{1}{2}} &= y''_n + h \left(\Psi_{250}f_{n-\frac{1}{6}} + \Psi_{251}f_{n-\frac{1}{3}} + \Psi_{252}f_n + \Psi_{253}f_{n+\frac{1}{6}} + \Psi_{254}f_{n+\frac{1}{3}} + \Psi_{255}f_{n+\frac{1}{2}} + \Psi_{256}f_{n+\frac{2}{3}} + \Psi_{257}f_{n+\frac{5}{6}} + \Psi_{258}f_{n+1} \right) \\
 y''_{n+\frac{2}{3}} &= y''_n + h \left(\Psi_{260}f_{n-\frac{1}{6}} + \Psi_{261}f_{n-\frac{1}{3}} + \Psi_{262}f_n + \Psi_{263}f_{n+\frac{1}{6}} + \Psi_{264}f_{n+\frac{1}{3}} + \Psi_{265}f_{n+\frac{1}{2}} + \Psi_{266}f_{n+\frac{2}{3}} + \Psi_{267}f_{n+\frac{5}{6}} + \Psi_{268}f_{n+1} \right) \\
 y''_{n+\frac{5}{6}} &= y''_n + h \left(\Psi_{270}f_{n-\frac{1}{6}} + \Psi_{271}f_{n-\frac{1}{3}} + \Psi_{272}f_n + \Psi_{273}f_{n+\frac{1}{6}} + \Psi_{274}f_{n+\frac{1}{3}} + \Psi_{275}f_{n+\frac{1}{2}} + \Psi_{276}f_{n+\frac{2}{3}} + \Psi_{277}f_{n+\frac{5}{6}} + \Psi_{278}f_{n+1} \right) \\
 y''_{n+1} &= y''_n + h \left(\Psi_{280}f_{n-\frac{1}{6}} + \Psi_{281}f_{n-\frac{1}{3}} + \Psi_{282}f_n + \Psi_{283}f_{n+\frac{1}{6}} + \Psi_{284}f_{n+\frac{1}{3}} + \Psi_{285}f_{n+\frac{1}{2}} + \Psi_{286}f_{n+\frac{2}{3}} + \Psi_{287}f_{n+\frac{5}{6}} + \Psi_{288}f_{n+1} \right)
 \end{aligned} \right\} \tag{8}$$

Therefore, to obtain the unknown coefficients of Π in (6), we use $\Pi_{i,j} = X^{-1}p$, that is

$$\begin{pmatrix} \Pi_{10} \\ \Pi_{11} \\ \Pi_{12} \\ \Pi_{13} \\ \Pi_{14} \\ \Pi_{15} \\ \Pi_{16} \\ \Pi_{17} \\ \Pi_{18} \end{pmatrix} = \begin{pmatrix} \frac{30887}{287400960} \\ \frac{5657}{7240063} \\ \frac{1231718400}{8622028800} \\ \frac{1308577}{4311014400} \\ \frac{124939}{574801920} \\ \frac{526213}{4311014400} \\ \frac{81113}{1724405760} \\ \frac{15713}{1437004800} \\ \frac{4967}{4311014400} \end{pmatrix}, \begin{pmatrix} \Pi_{20} \\ \Pi_{21} \\ \Pi_{22} \\ \Pi_{23} \\ \Pi_{24} \\ \Pi_{25} \\ \Pi_{26} \\ \Pi_{27} \\ \Pi_{28} \end{pmatrix} = \begin{pmatrix} \frac{83459}{33679800} \\ \frac{997}{156677} \\ \frac{44906400}{33679800} \\ \frac{755}{449064} \\ \frac{7603}{6735960} \\ \frac{20479}{33679800} \\ \frac{2557}{11226600} \\ \frac{1747}{33679800} \\ \frac{29}{5388768} \end{pmatrix}, \begin{pmatrix} \Pi_{30} \\ \Pi_{31} \\ \Pi_{32} \\ \Pi_{33} \\ \Pi_{34} \\ \Pi_{35} \\ \Pi_{36} \\ \Pi_{37} \\ \Pi_{38} \end{pmatrix} = \begin{pmatrix} \frac{2873}{123171840} \\ \frac{589}{359251200} \\ \frac{4793773}{8622028800} \\ \frac{69823}{205286400} \\ \frac{54863}{344881152} \\ \frac{341687}{4311014400} \\ \frac{16559}{574801920} \\ \frac{3983}{615859200} \\ \frac{5741}{8622028800} \end{pmatrix}, \begin{pmatrix} \Pi_{40} \\ \Pi_{41} \\ \Pi_{42} \\ \Pi_{43} \\ \Pi_{44} \\ \Pi_{45} \\ \Pi_{46} \\ \Pi_{47} \\ \Pi_{48} \end{pmatrix} = \begin{pmatrix} \frac{1697}{11226600} \\ \frac{1451}{134719200} \\ \frac{101237}{33679800} \\ \frac{26317}{6735960} \\ \frac{31}{32076} \\ \frac{17471}{33679800} \\ \frac{6409}{33679800} \\ \frac{481}{11226600} \\ \frac{17}{3849120} \end{pmatrix},$$

$$\begin{pmatrix} \Pi_{50} \\ \Pi_{51} \\ \Pi_{52} \\ \Pi_{53} \\ \Pi_{54} \\ \Pi_{55} \\ \Pi_{56} \\ \Pi_{57} \\ \Pi_{58} \end{pmatrix} = \begin{pmatrix} \frac{731}{1971200} \\ \frac{103}{3942400} \\ \frac{28899}{3942400} \\ \frac{5119}{394240} \\ \frac{1}{788480} \\ \frac{1019}{844800} \\ \frac{1703}{3942400} \\ \frac{193}{1971200} \\ \frac{1}{98560} \end{pmatrix}, \begin{pmatrix} \Pi_{60} \\ \Pi_{61} \\ \Pi_{62} \\ \Pi_{63} \\ \Pi_{64} \\ \Pi_{65} \\ \Pi_{66} \\ \Pi_{67} \\ \Pi_{68} \end{pmatrix} = \begin{pmatrix} \frac{584}{841995} \\ \frac{137}{2806650} \\ \frac{57179}{4209975} \\ \frac{38536}{1403325} \\ \frac{4351}{841995} \\ \frac{18632}{4209975} \\ \frac{31}{4209975} \\ \frac{40095}{776} \\ \frac{23}{1202850} \end{pmatrix}, \begin{pmatrix} \Pi_{70} \\ \Pi_{71} \\ \Pi_{72} \\ \Pi_{73} \\ \Pi_{74} \\ \Pi_{75} \\ \Pi_{76} \\ \Pi_{77} \\ \Pi_{78} \end{pmatrix} = \begin{pmatrix} \frac{63625}{57480192} \\ \frac{13375}{172440576} \\ \frac{7485125}{344881152} \\ \frac{8189875}{172440576} \\ \frac{1639375}{114960384} \\ \frac{2200375}{172440576} \\ \frac{332375}{344881152} \\ \frac{18875}{57480192} \\ \frac{10375}{344881152} \end{pmatrix}, \begin{pmatrix} \Pi_{80} \\ \Pi_{81} \\ \Pi_{82} \\ \Pi_{83} \\ \Pi_{84} \\ \Pi_{85} \\ \Pi_{86} \\ \Pi_{87} \\ \Pi_{88} \end{pmatrix} = \begin{pmatrix} \frac{1}{616} \\ \frac{1}{8800} \\ \frac{489}{15400} \\ \frac{1123}{15400} \\ \frac{17}{616} \\ \frac{397}{15400} \\ \frac{23}{3080} \\ \frac{41}{15400} \\ \frac{1}{184800} \end{pmatrix}$$

Similarly, to obtain the unknown coefficients of higher derivatives Ψ in (7) and (8), we consider $\Psi_{rja} = X^{-1}Z$ that.

$$\begin{array}{c}
 \left(\begin{array}{c} \Psi_{110} \\ \Psi_{111} \\ \Psi_{112} \\ \Psi_{113} \\ \Psi_{114} \\ \Psi_{115} \\ \Psi_{116} \\ \Psi_{117} \\ \Psi_{118} \end{array} \right) = \left(\begin{array}{c} 90019 \\ 4665600 \\ 698039 \\ 261273600 \\ 502777 \\ 5443200 \\ 2643601 \\ 16329600 \\ 5201299 \\ 26127360 \\ 573997 \\ 3628800 \\ 2573323 \\ 32659200 \\ 183373 \\ 8164800 \\ 34871 \\ 12441600 \end{array} \right), \left(\begin{array}{c} \Psi_{120} \\ \Psi_{121} \\ \Psi_{122} \\ \Psi_{123} \\ \Psi_{124} \\ \Psi_{125} \\ \Psi_{126} \\ \Psi_{127} \\ \Psi_{128} \end{array} \right) = \left(\begin{array}{c} 5797933 \\ 255150 \\ 11618689 \\ 4082400 \\ 3387899 \\ 42525 \\ 8128579 \\ 51030 \\ 16256875 \\ 81648 \\ 13547353 \\ 85050 \\ 20321023 \\ 255150 \\ 5806009 \\ 255150 \\ 17203 \\ 6048 \end{array} \right), \left(\begin{array}{c} \Psi_{130} \\ \Psi_{131} \\ \Psi_{132} \\ \Psi_{133} \\ \Psi_{134} \\ \Psi_{135} \\ \Psi_{136} \\ \Psi_{137} \\ \Psi_{138} \end{array} \right) = \left(\begin{array}{c} 14797 \\ 653184 \\ 733819 \\ 261273600 \\ 941567 \\ 10886400 \\ 690037 \\ 4665600 \\ 4996883 \\ 26127360 \\ 419099 \\ 2721600 \\ 25217 \\ 326592 \\ 721631 \\ 32659200 \\ 80263 \\ 29030400 \end{array} \right), \left(\begin{array}{c} \Psi_{140} \\ \Psi_{141} \\ \Psi_{142} \\ \Psi_{143} \\ \Psi_{144} \\ \Psi_{145} \\ \Psi_{146} \\ \Psi_{147} \\ \Psi_{148} \end{array} \right) = \left(\begin{array}{c} 5806331 \\ 255150 \\ 11612429 \\ 4082400 \\ 2710189 \\ 34020 \\ 40632853 \\ 255150 \\ 11611921 \\ 58320 \\ 501749 \\ 3150 \\ 40641847 \\ 510300 \\ 1161199 \\ 51030 \\ 552953 \\ 194400 \end{array} \right), \left(\begin{array}{c} \Psi_{150} \\ \Psi_{151} \\ \Psi_{152} \\ \Psi_{153} \\ \Psi_{154} \\ \Psi_{155} \\ \Psi_{156} \\ \Psi_{157} \\ \Psi_{158} \end{array} \right) = \left(\begin{array}{c} 58786471 \\ 44800 \\ 58786439 \\ 358400 \\ 154315361 \\ 33600 \\ 20575081 \\ 2240 \\ 82301087 \\ 7168 \\ 58786349 \\ 6400 \\ 205752301 \\ 44800 \\ 4199027 \\ 3200 \\ 35271829 \\ 215040 \end{array} \right), \left(\begin{array}{c} \Psi_{160} \\ \Psi_{161} \\ \Psi_{162} \\ \Psi_{163} \\ \Psi_{164} \\ \Psi_{165} \\ \Psi_{166} \\ \Psi_{167} \\ \Psi_{168} \end{array} \right) = \left(\begin{array}{c} 2972705048 \\ 127575 \\ 106168033 \\ 36450 \\ 3468157394 \\ 42525 \\ 2080892015 \\ 2 \\ 127575 \\ 5202234431 \\ 25515 \\ 6936309656 \\ 42525 \\ 1486352384 \\ 18225 \\ 2972704712 \\ 127575 \\ 82575131 \\ 28350 \end{array} \right), \\
 \\
 \left(\begin{array}{c} \Psi_{170} \\ \Psi_{171} \\ \Psi_{172} \\ \Psi_{173} \\ \Psi_{174} \\ \Psi_{175} \\ \Psi_{176} \\ \Psi_{177} \\ \Psi_{178} \end{array} \right) = \left(\begin{array}{c} 7087480562 \\ 5 \\ 326592 \\ 2834992208 \\ 75 \\ 10450944 \\ 3307491124 \\ 25 \\ 435456 \\ 1984494354 \\ 125 \\ 1306368 \\ 9922473014 \\ 375 \\ 5225472 \\ 1102496916 \\ 25 \\ 72576 \\ 4961236493 \\ 75 \\ 653184 \\ 2834992149 \\ 25 \\ 1306368 \\ 9449973912 \\ 5 \\ 3483648 \end{array} \right), \left(\begin{array}{c} \Psi_{180} \\ \Psi_{181} \\ \Psi_{182} \\ \Psi_{183} \\ \Psi_{184} \\ \Psi_{185} \\ \Psi_{186} \\ \Psi_{187} \\ \Psi_{188} \end{array} \right) = \left(\begin{array}{c} 13436891 \\ 10 \\ 940582369 \\ 5600 \\ 4938057499 \\ 1050 \\ 3292038227 \\ 350 \\ 6584076623 \\ 560 \\ 9876114757 \\ 1050 \\ 164601916 \\ 35 \\ 470291173 \\ 350 \\ 134368911 \\ 800 \end{array} \right), \left(\begin{array}{c} \Psi_{210} \\ \Psi_{211} \\ \Psi_{212} \\ \Psi_{213} \\ \Psi_{214} \\ \Psi_{215} \\ \Psi_{216} \\ \Psi_{217} \\ \Psi_{218} \end{array} \right) = \left(\begin{array}{c} 308321 \\ 3628800 \\ 123079 \\ 7257600 \\ 2422351 \\ 3628800 \\ 4055197 \\ 3628800 \\ 492343 \\ 362880 \\ 3888947 \\ 3628800 \\ 1930601 \\ 3628800 \\ 549071 \\ 3628800 \\ 19547 \\ 1036800 \end{array} \right), \left(\begin{array}{c} \Psi_{220} \\ \Psi_{221} \\ \Psi_{222} \\ \Psi_{223} \\ \Psi_{224} \\ \Psi_{225} \\ \Psi_{226} \\ \Psi_{227} \\ \Psi_{228} \end{array} \right) = \left(\begin{array}{c} 2142739 \\ 28350 \\ 2161139 \\ 226800 \\ 30097771 \\ 113400 \\ 15047423 \\ 28350 \\ 3762139 \\ 5670 \\ 15049313 \\ 28350 \\ 30099661 \\ 113400 \\ 307147 \\ 4050 \\ 2150069 \\ 226800 \end{array} \right), \left(\begin{array}{c} \Psi_{230} \\ \Psi_{231} \\ \Psi_{232} \\ \Psi_{233} \\ \Psi_{234} \\ \Psi_{235} \\ \Psi_{236} \\ \Psi_{237} \\ \Psi_{238} \end{array} \right) = \left(\begin{array}{c} 554191 \\ 3628800 \\ 19547 \\ 1036800 \\ 2154601 \\ 3628800 \\ 3324467 \\ 3628800 \\ 456503 \\ 362880 \\ 3696797 \\ 3628800 \\ 1857871 \\ 3628800 \\ 532321 \\ 3628800 \\ 133319 \\ 7257600 \end{array} \right), \\
 \\
 \left(\begin{array}{c} \Psi_{250} \\ \Psi_{251} \\ \Psi_{252} \\ \Psi_{253} \\ \Psi_{254} \\ \Psi_{255} \\ \Psi_{256} \\ \Psi_{256} \\ \Psi_{258} \end{array} \right) = \left(\begin{array}{c} 130633743 \\ 44800 \\ 32658417 \\ 89600 \\ 1371661939 \\ 134400 \\ 914426451 \\ 44800 \\ 114305109 \\ 4480 \\ 2743293143 \\ 134400 \\ 457216983 \\ 44800 \\ 130633473 \\ 44800 \\ 97975121 \\ 268800 \end{array} \right), \left(\begin{array}{c} \Psi_{260} \\ \Psi_{261} \\ \Psi_{262} \\ \Psi_{263} \\ \Psi_{264} \\ \Psi_{265} \\ \Psi_{266} \\ \Psi_{267} \\ \Psi_{268} \end{array} \right) = \left(\begin{array}{c} 550488784 \\ 14175 \\ 19660313 \\ 4050 \\ 1926711529 \\ 14175 \\ 3853418288 \\ 14175 \\ 963355669 \\ 2835 \\ 3853418288 \\ 14175 \\ 1926711529 \\ 14175 \\ 550488784 \\ 14175 \\ 19660313 \\ 4050 \end{array} \right), \left(\begin{array}{c} \Psi_{270} \\ \Psi_{271} \\ \Psi_{272} \\ \Psi_{273} \\ \Psi_{274} \\ \Psi_{275} \\ \Psi_{276} \\ \Psi_{277} \\ \Psi_{278} \end{array} \right) = \left(\begin{array}{c} 4199895917 \\ 5 \\ 145152 \\ 1049973972 \\ 5 \\ 290304 \\ 1469963655 \\ 05 \\ 145152 \\ 2939926832 \\ 75 \\ 145152 \\ 1837454563 \\ 75 \\ 72576 \\ 2939926899 \\ 25 \\ 145152 \\ 1469963851 \\ 75 \\ 145152 \\ 5999849935 \\ 20736 \\ 1049973917 \\ 5 \\ 290304 \end{array} \right), \left(\begin{array}{c} \Psi_{280} \\ \Psi_{281} \\ \Psi_{282} \\ \Psi_{283} \\ \Psi_{284} \\ \Psi_{285} \\ \Psi_{286} \\ \Psi_{287} \\ \Psi_{288} \end{array} \right) = \left(\begin{array}{c} 522534237 \\ 350 \\ 522534237 \\ 2800 \\ 2194643815 \\ 9 \\ 4200 \\ 3657739569 \\ 350 \\ 914434917 \\ 70 \\ 1097321863 \\ 7 \\ 1050 \\ 7315479363 \\ 1400 \\ 522534147 \\ 350 \\ 223943303 \\ 1200 \end{array} \right), \left(\begin{array}{c} \Psi_{240} \\ \Psi_{241} \\ \Psi_{242} \\ \Psi_{243} \\ \Psi_{244} \\ \Psi_{245} \\ \Psi_{246} \\ \Psi_{247} \\ \Psi_{248} \end{array} \right) = \left(\begin{array}{c} 2150429 \\ 28350 \\ 2150389 \\ 226800 \\ 30112261 \\ 113400 \\ 15046513 \\ 28350 \\ 3763469 \\ 5670 \\ 15052463 \\ 28350 \\ 30104771 \\ 113400 \\ 2150339 \\ 28350 \\ 2150339 \\ 226800 \end{array} \right)
 \end{array}$$

3. Analysis of Basic Properties Numerical Scheme

The analysis of the basic properties of the NNS was conducted. These properties are order and error constant, consistency, zero-stability, convergent, and region of absolute stability.

3.1. Order and Error Constant

We consider the linear operator $L[y(t_n);h]$ with the following corollaries 1 and 2 to determine the order and error constant of the new method.

3.1.1. Corollary 1

The linear operator $L[y(t_n);h]$ associated with the local truncation error of the NNS (6) and its higher derivatives (7) and (8) is

$$C_{10}h^{10}y^{(10)}(t_n)+O(h^{13}), C_{09}h^{09}y^{(09)}(t_n)+O(h^{11}), C_{09}h^{09}y^{(09)}(t_n)+O(h^{11}) \quad [20]$$

Proof

According to [20], the linear difference operators associated with the NNS are given by

$$\left. \begin{aligned} L[y(\xi_n);h] &= y_{\frac{n-1}{6}} - y_n + \frac{1}{6}hy'_n - \frac{\left(\frac{1}{6}h\right)^2}{2!}y''_n - h^3\left(\Pi_{10}f_{\frac{n-1}{6}} + \Pi_{11}f_{\frac{n-1}{3}} + \Pi_{12}f_n + \Pi_{13}f_{\frac{n+1}{6}} + \Pi_{14}f_{\frac{n+1}{3}} + \Pi_{15}f_{\frac{n+1}{2}} + \Pi_{16}f_{\frac{n+2}{3}} + \Pi_{17}f_{\frac{n+5}{6}} + \Pi_{18}f_{n+1}\right) \\ L[y(\xi_n);h] &= y_{\frac{n-1}{3}} - y_n + \frac{1}{3}hy'_n - \frac{\left(\frac{1}{3}h\right)^2}{2!}y''_n - h^3\left(\Pi_{20}f_{\frac{n-1}{6}} + \Pi_{21}f_{\frac{n-1}{3}} + \Pi_{22}f_n + \Pi_{23}f_{\frac{n+1}{6}} + \Pi_{24}f_{\frac{n+1}{3}} + \Pi_{25}f_{\frac{n+1}{2}} + \Pi_{26}f_{\frac{n+2}{3}} + \Pi_{27}f_{\frac{n+5}{6}} + \Pi_{28}f_{n+1}\right) \\ L[y(\xi_n);h] &= y_{\frac{n+1}{6}} - y_n - \frac{1}{6}hy'_n - \frac{\left(\frac{1}{6}h\right)^2}{2!}y''_n - h^3\left(\Pi_{30}f_{\frac{n-1}{6}} + \Pi_{31}f_{\frac{n-1}{3}} + \Pi_{32}f_n + \Pi_{33}f_{\frac{n+1}{6}} + \Pi_{34}f_{\frac{n+1}{3}} + \Pi_{35}f_{\frac{n+1}{2}} + \Pi_{36}f_{\frac{n+2}{3}} + \Pi_{37}f_{\frac{n+5}{6}} + \Pi_{38}f_{n+1}\right) \\ L[y(\xi_n);h] &= y_{\frac{n+1}{3}} - y_n - \frac{1}{3}hy'_n - \frac{\left(\frac{1}{3}h\right)^2}{2!}y''_n - h^3\left(\Pi_{40}f_{\frac{n-1}{6}} + \Pi_{41}f_{\frac{n-1}{3}} + \Pi_{42}f_n + \Pi_{43}f_{\frac{n+1}{6}} + \Pi_{44}f_{\frac{n+1}{3}} + \Pi_{45}f_{\frac{n+1}{2}} + \Pi_{46}f_{\frac{n+2}{3}} + \Pi_{47}f_{\frac{n+5}{6}} + \Pi_{48}f_{n+1}\right) \\ L[y(\xi_n);h] &= y_{\frac{n+1}{2}} - y_n - \frac{1}{2}hy'_n - \frac{\left(\frac{1}{2}h\right)^2}{2!}y''_n - h^3\left(\Pi_{50}f_{\frac{n-1}{6}} + \Pi_{51}f_{\frac{n-1}{3}} + \Pi_{52}f_n + \Pi_{53}f_{\frac{n+1}{6}} + \Pi_{54}f_{\frac{n+1}{3}} + \Pi_{55}f_{\frac{n+1}{2}} + \Pi_{56}f_{\frac{n+2}{3}} + \Pi_{57}f_{\frac{n+5}{6}} + \Pi_{58}f_{n+1}\right) \\ L[y(\xi_n);h] &= y_{\frac{n+2}{3}} - y_n - \frac{2}{3}hy'_n - \frac{\left(\frac{2}{3}h\right)^2}{2!}y''_n - h^3\left(\Pi_{60}f_{\frac{n-1}{6}} + \Pi_{61}f_{\frac{n-1}{3}} + \Pi_{62}f_n + \Pi_{63}f_{\frac{n+1}{6}} + \Pi_{64}f_{\frac{n+1}{3}} + \Pi_{65}f_{\frac{n+1}{2}} + \Pi_{66}f_{\frac{n+2}{3}} + \Pi_{67}f_{\frac{n+5}{6}} + \Pi_{68}f_{n+1}\right) \\ L[y(\xi_n);h] &= y_{\frac{n+5}{6}} - y_n - \frac{5}{6}hy'_n - \frac{\left(\frac{5}{6}h\right)^2}{2!}y''_n - h^3\left(\Pi_{70}f_{\frac{n-1}{6}} + \Pi_{71}f_{\frac{n-1}{3}} + \Pi_{72}f_n + \Pi_{73}f_{\frac{n+1}{6}} + \Pi_{74}f_{\frac{n+1}{3}} + \Pi_{75}f_{\frac{n+1}{2}} + \Pi_{76}f_{\frac{n+2}{3}} + \Pi_{77}f_{\frac{n+5}{6}} + \Pi_{78}f_{n+1}\right) \\ L[y(\xi_n);h] &= y_{n+1} - y_n - y'_n - \frac{(h)^2}{2!}y''_n - h^3\left(\Pi_{80}f_{\frac{n-1}{6}} + \Pi_{81}f_{\frac{n-1}{3}} + \Pi_{82}f_n + \Pi_{83}f_{\frac{n+1}{6}} + \Pi_{84}f_{\frac{n+1}{3}} + \Pi_{85}f_{\frac{n+1}{2}} + \Pi_{86}f_{\frac{n+2}{3}} + \Pi_{87}f_{\frac{n+5}{6}} + \Pi_{88}f_{n+1}\right) \end{aligned} \right\} \quad (9)$$

3.1.2. Corollary 2 [20]

The local truncation error of NNS is assumed $y(x)$ to be sufficiently differentiable, and expanding Equation (9) to (11) by x_n using the Taylor series to have

$$\begin{aligned}
 L_{\frac{1}{6}}[y(x_n);h] &= (1.1025 \times 10^{-08}), L_{\frac{1}{3}}[y(x_n);h] = (5.6449 \times 10^{-06}), L_{\frac{1}{6}}[y(x_n);h] = (-1.1025 \times 10^{-08}), L_{\frac{1}{3}}[y(x_n);h] = (-5.6449 \times 10^{-06}), \\
 L_{\frac{1}{2}}[y(x_n);h] &= (-2.1701 \times 10^{-04}), L_{\frac{2}{3}}[y(x_n);h] = (-2.8902 \times 10^{-03}), L_{\frac{5}{6}}[y(x_n);h] = (-2.1534 \times 10^{-02}), L_1[y(x_n);h] = (-1.1111 \times 10^{-01}), \\
 L_{\frac{1}{6}}[y'(x_n);h] &= (-1.6538 \times 10^{-09}), L_{\frac{1}{3}}[y'(x_n);h] = (-1.6935 \times 10^{-06}), L_{\frac{1}{6}}[y'(x_n);h] = (-1.6538 \times 10^{-09}), L_{\frac{1}{3}}[y'(x_n);h] = (-1.6935 \times 10^{-06}), \\
 L_{\frac{1}{2}}[y'(x_n);h] &= (-9.7656 \times 10^{-05}), L_{\frac{2}{3}}[y'(x_n);h] = (-1.7341 \times 10^{-03}), L_{\frac{5}{6}}[y'(x_n);h] = (-1.6151 \times 10^{-02}), L_1[y'(x_n);h] = (-1.0000 \times 10^{-01}), \\
 L_{\frac{1}{6}}[y''(x_n);h] &= (8.0967 \times 10^{-14}), L_{\frac{1}{3}}[y''(x_n);h] = (3.3870 \times 10^{-13}), L_{\frac{1}{6}}[y''(x_n);h] = (4.1676 \times 10^{-14}), L_{\frac{1}{3}}[y''(x_n);h] = (2.7682 \times 10^{-13}), \\
 L_{\frac{1}{2}}[y''(x_n);h] &= (6.5057 \times 10^{-13}), L_{\frac{2}{3}}[y''(x_n);h] = (1.2178 \times 10^{-12}), L_{\frac{5}{6}}[y''(x_n);h] = (1.9076 \times 10^{-12}), L_1[y''(x_n);h] = (2.7743 \times 10^{-12}),
 \end{aligned}$$

Proof

Expand equations (9) to (11) using Corollary 2 and then collect the like terms to the power of h gives

$$\begin{aligned}
 L_{\frac{1}{6}}[y(x_n);h] &= (1.1025 \times 10^{-08})C_{10}h^{10}y^{10}(t_n) + O(h^{13}), L_{\frac{1}{3}}[y(x_n);h] = (5.6449 \times 10^{-06})C_{10}h^{10}y^{10}(t_n) + O(h^{13}) \\
 L_{\frac{1}{6}}[y(x_n);h] &= (-1.1025 \times 10^{-08})C_{10}h^{10}y^{10}(t_n) + O(h^{13}), L_{\frac{1}{3}}[y(x_n);h] = (-5.6449 \times 10^{-06})C_{10}h^{10}y^{10}(t_n) + O(h^{13}) \\
 L_{\frac{1}{2}}[y(x_n);h] &= (-2.1701 \times 10^{-04})C_{10}h^{10}y^{10}(t_n) + O(h^{13}), L_{\frac{2}{3}}[y(x_n);h] = (-2.8902 \times 10^{-03})C_{10}h^{10}y^{10}(t_n) + O(h^{13}) \\
 L_{\frac{5}{6}}[y(x_n);h] &= (-2.1534 \times 10^{-02})C_{10}h^{10}y^{10}(t_n) + O(h^{13}), L_1[y(x_n);h] = (-1.1111 \times 10^{-01})C_{10}h^{10}y^{10}(t_n) + O(h^{13}) \\
 L_{\frac{1}{6}}[y'(x_n);h] &= (-1.6538 \times 10^{-09})C_{09}h^{09}y^{09}(t_n) + O(h^{11}), L_{\frac{1}{3}}[y'(x_n);h] = (-1.6935 \times 10^{-06})C_{09}h^{09}y^{09}(t_n) + O(h^{11}) \\
 L_{\frac{1}{6}}[y'(x_n);h] &= (-1.6538 \times 10^{-09})C_{09}h^{09}y^{09}(t_n) + O(h^{11}), L_{\frac{1}{3}}[y'(x_n);h] = (-1.6935 \times 10^{-06})C_{09}h^{09}y^{09}(t_n) + O(h^{11}) \\
 L_{\frac{1}{2}}[y'(x_n);h] &= (-9.7656 \times 10^{-05})C_{09}h^{09}y^{09}(t_n) + O(h^{11}), L_{\frac{2}{3}}[y'(x_n);h] = (-1.7341 \times 10^{-03})C_{09}h^{09}y^{09}(t_n) + O(h^{11}) \\
 L_{\frac{5}{6}}[y'(x_n);h] &= (-1.6151 \times 10^{-02})C_{09}h^{09}y^{09}(t_n) + O(h^{11}), L_1[y'(x_n);h] = (-1.0000 \times 10^{-01})C_{09}h^{09}y^{09}(t_n) + O(h^{11}) \\
 L_{\frac{1}{6}}[y''(x_n);h] &= (8.0967 \times 10^{-14})C_{09}h^{09}y^{09}(t_n) + O(h^{10}), L_{\frac{1}{3}}[y''(x_n);h] = (3.3870 \times 10^{-13})C_{09}h^{09}y^{09}(t_n) + O(h^{10}) \\
 L_{\frac{1}{6}}[y''(x_n);h] &= (4.1676 \times 10^{-14})C_{09}h^{09}y^{09}(t_n) + O(h^{10}), L_{\frac{1}{3}}[y''(x_n);h] = (2.7682 \times 10^{-13})C_{09}h^{09}y^{09}(t_n) + O(h^{10}) \\
 L_{\frac{1}{2}}[y''(x_n);h] &= (6.5057 \times 10^{-13})C_{09}h^{09}y^{09}(t_n) + O(h^{10}), L_{\frac{2}{3}}[y''(x_n);h] = (1.2178 \times 10^{-12})C_{09}h^{09}y^{09}(t_n) + O(h^{10}) \\
 L_{\frac{5}{6}}[y''(x_n);h] &= (1.9076 \times 10^{-12})C_{09}h^{09}y^{09}(t_n) + O(h^{10}), L_1[y''(x_n);h] = (2.7743 \times 10^{-12})C_{09}h^{09}y^{09}(t_n) + O(h^{10})
 \end{aligned}$$

3.2. Consistency

According to [21], a NNS is said to be consistent if it has an order of convergence greater than or equal to zero, i.e., ($p \geq 1$). Thus, NNS are consistent.

3.3. Zero Stability

An NNS is said to be Zero-stable for any well-behaved initial value problem, provided that

- All roots of $y(r)$ lies in the unit disk, $|r| \leq 1$
- Any roots on the unit circle ($|r| = 1$) are simple [20].

Hence,

$$\rho(z) = z^8 - \frac{15232}{350} z^7 + \frac{18124}{334} z^6 - \frac{22387}{13} z^5 + \frac{2734}{2} z^4 - 26508 z^3 + 171328 z^2 + 7549742 z + 150994944 \tag{12}$$

Now set (12) equal to zero and solving for z gives $z=1$, hence the method is zero stable.

3.4. Convergence

According to [21], the necessary and sufficient condition for an NNS to be convergent is that it must be consistent and zero stable. Since the NNS is consistent and zero stable, it is convergent.

3.5. Region of Absolute Stability

3.5.1. Definition 3.3

The region of absolute stability of NNS is the set of complex values λh for which all solutions of the test problem $y''' = -\lambda^3 y$ will remain bounded as $n \rightarrow \infty$.

The concept of A-stability, according to [20], is discussed by applying the test equation

$$y^{(k)} = \lambda^{(k)} y \tag{13}$$

to yield

$$Y_m = \lambda(x) Y_{m-1}, x = \lambda h \tag{14}$$

Where $\lambda(x)$ is the amplification matrix of the form

$$\lambda(x) = (v^0 - xt^{(0)} - x^4 t^{(0)})^{-1} (\xi^1 - xt^{(1)} - x^4 t^{(1)}) \tag{15}$$

The matrix $\mu(z)$ has Eigenvalues $(0, 0, \Lambda, \nu_k)$, which ν_k are called the stability function.

The boundary locus method is adopted in generating the stability polynomial of the hybrid method. The polynomial is

$$\bar{h}(w) = \left\{ \begin{aligned} &\left(\frac{643529}{16533720} w^7 + \frac{5}{52488} w^8 \right) h^8 + \left(-\frac{19}{17496} w^8 + \frac{2574431}{5511240} w^7 \right) h^7 + \left(\frac{1295173}{262440} w^7 - \frac{331}{17496} w^8 \right) h^6 + \left(\frac{91}{162} w^8 + \frac{326647}{7290} w^7 \right) h^5 + \\ &\left(\frac{162431}{486} w^7 - \frac{385}{162} w^8 \right) h^4 + \left(-\frac{280}{3} w^8 + \frac{153716}{81} w^7 \right) h^3 + \left(\frac{64678}{9} w^7 - \frac{5320}{3} w^8 \right) h^2 + \left(-13440 w^8 + \frac{40318}{40318} w^7 \right) h - w^7 + w^8 \end{aligned} \right\} \tag{16}$$

The polynomial (16) is used to plot the region as

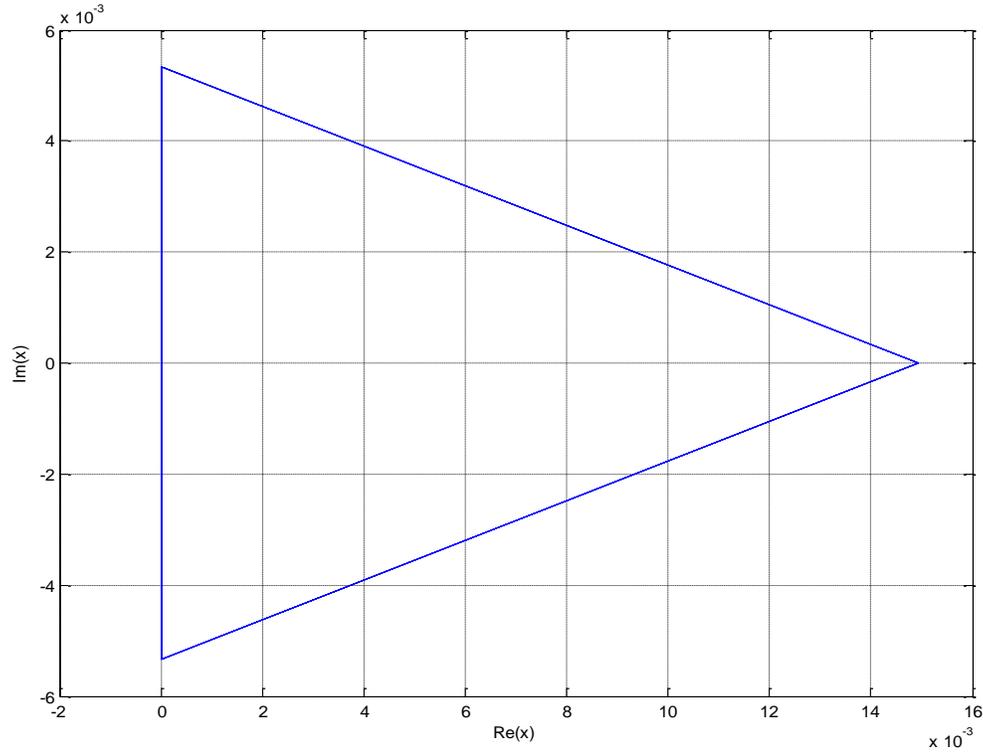


Fig. 1 Regions of absolute stability of the NNS

4. Numerical Examples

This section deals with the numerical presentation of the performance of the new scheme for numerical integrations of the Volterra Integro-Differential Equation of the second kind. The absolute error or maximum error in the new scheme is computed and compared with the existing methods in the literature.

4.1. Example 1

Consider the Volterra Integro-Differential Equation of the form

$$y'(x) = 1 - \int_0^x y(t)dt, \quad y(0) = 0, \quad 0 \leq x \leq 1 \tag{17}$$

with the exact solution given by

$$y(x) = \sin(x) \tag{18}$$

Example 1 was solved using various step sizes. Table 1 shows the comparison of the New Numerical Scheme (NNS) when $h=0.01$ with the existing methods (NETM) as in [18], (TM), and (ETM) in [22]. Table 2 also shows the comparison of NNS $h = \frac{1}{2^i}, i = 3(1)7$ with existing methods (ABM5) and (2P3SBMVIDE) in [12] and (2P3BVIDE) in [23].

4.2. Example 2

Consider the Volterra integro-differential equation of the form

$$y'(x) = -\sin(x) - \cos(x) + \int_0^x 2 \cos(x-t)y(x)dt, y(0) = 1 \tag{19}$$

with the exact solution given by

$$y(x) = \exp(-x) \tag{20}$$

Table 3 presents the results of example 2. The Volterra integro-differential equation was solved using the new NNS. The numerical results in comparison with the (ABM5) and Two Point three-step block method (2P3BVIDE) are both as given in [23]. The results are also compared with the BTFM of [24].

4.3. Example 3

Consider the nonlinear Volterra integro-differential equation of the form

$$y''(x) + \int_0^x (y(t))^2 + \left(\frac{x}{2} - \sinh(x) - \frac{1}{4} \sinh(2x)\right) = 0, y(0) = 0, y'(0) = 1 \tag{21}$$

with the exact solution given by

$$y(x) = \sinh(x) \tag{22}$$

Table 4 shows the comparison example 3, which was solved using the NNS, and the results are compared with those given by the CMM proposed by [25]. The results are also compared to a (FDTFBBDF) proposed by [26] and methods (4SBM and 5SBM) proposed by [27].

Table 1. The numerical results for example 1 (ENNS compare with ETM, EETM, and ENETM)

<i>x</i>	Exact Solution	Computed Solution	ENNS	ETM	EETM	ENETM
1	0.00999983333416666468	0.00999983333416666573	1.0500(-18)	8.2978(-07)	8.2916(-07)	1.2051(-09)
2	0.01999866669333307937	0.01999866669333307752	1.8500(-18)	1.6359(-06)	1.6334(-06)	1.0290(-08)
3	0.02999550020249566077	0.02999550020249568805	2.7280(-17)	2.3938(-06)	2.3883(-06)	3.5392(-08)
4	0.03998933418663415945	0.03998933418663409131	6.8140(-17)	3.0799(-06)	3.0702(-06)	8.4328(-08)
5	0.04997916927067832880	0.04997916927067846282	1.3402(-16)	3.6715(-06)	3.6565(-06)	1.6443(-07)
6	0.05996400647944459920	0.05996400647944453973	5.9470(-17)	4.1478(-06)	4.1266(-06)	2.8242(-07)
7	0.06994284733753276398	0.06994284733753262117	1.4281(-16)	4.4897(-06)	4.4615(-06)	4.4421(-07)
8	0.07991469396917268731	0.07991469396917359916	9.1185(-16)	4.6805(-06)	4.6447(-06)	6.5484(-07)
9	0.08987854919801104969	0.08987854919801180634	7.5665(-16)	4.7061(-06)	4.6620(-06)	9.1831(-07)
10	0.09983341664682815231	0.09983341664682827558	1.2327(-16)	4.5551(-06)	4.5025(-06)	1.2375(-06)

Table 2. The numerical results for example 1 (ENNS compare with 2P3BVIDE, ABM5, and 2P3SBMVIDE)

x	Exact Solution	Computed Solution	ENNS	2P3BVIDE	ABM5	2P3SBMVIDE
0.025	0.02499739591471233066	0.02499739591471232708	3.5800(-18)	5.7323(-08)	2.8951(-07)	1.2349(-09)
0.0125	0.01249967448170978872	0.01249967448170978962	9.0000(-19)	5.5893(-09)	3.6127(-08)	3.8642(-11)
0.00625	0.00624995930997530612	0.00624995930997530597	1.5000(-19)	2.2443(-10)	4.3953(-09)	1.2080(-12)
0.003125	0.00312499491373946269	0.00312499491373946268	1.0000(-20)	1.3908(-11)	5.4213(-10)	3.7751(-14)
0.0015625	0.00156249936421720001	0.00156249936421720001	0.0000(00)	8.6930(-13)	6.7325(-11)	5.3291(-15)
0.00078125	0.00078124992052714272	0.00078124992052714272	0.0000(00)	-	-	1.3545(-14)

Table 3. The numerical results for example 2 (ENNS compare with ABM5, 2P3BVIDE, and BTFM)

x	Exact Solution	Computed Solution	ENNS	ABM5	2P3BVIDE	BTFM
0.250	0.77880078307140486825	0.77880078307140540806	5.3981(-16)	8.1337(-03)	6.1138(-03)	1.6867(-06)
0.1250	0.88249690258459540286	0.88249690258459540322	3.6000(-19)	4.7616(-04)	3.9009(-04)	5.0991(-08)
0.06250	0.93941306281347578612	0.93941306281347578612	0.0000(00)	2.1034(-05)	1.6881(-05)	1.5744(-09)
0.031250	0.96923323447634408185	0.96923323447634408185	0.0000(00)	7.8509(-07)	7.8509(-07)	4.8948(-11)
0.0156250	0.98449643700540840599	0.98449643700540840598	1.0000(-20)	2.6828(-08)	2.0516(-08)	1.5261(-12)
0.00781250	0.99221793826024351211	0.99221793826024351211	0.0000(00)	8.7684(-10)	6.6334(-10)	4.7637(-14)

Table 4. The numerical results for example 3 (ENNS compare with CMM, FDTFBBDF, 4SBM, and 5SBM)

x	Exact Solution	Computed Solution	ENNS	CMM	FDTFBBDF	4SBM	5SBM
0.16	0.16068354101279944828	0.16068354101279129786	8.1504(-15)	4.5700(-08)	7.8800(-05)	4.5800(-08)	4.5700(-08)
0.32	0.32548936363113307984	0.32548936363113985654	6.7767(-15)	3.7140(-07)	2.9800(-04)	3.7160(-07)	3.7150(-07)
0.48	0.49864550519337626463	0.49864550519337347295	2.7917(-15)	1.2858e(06)	6.5900(-04)	1.2861(-06)	1.2859(-06)
0.64	0.68459422763095139805	0.68459422763099268709	4.1289(-14)	3.1554(-06)	1.1900(-03)	3.1559(-06)	3.1555(-06)
0.80	0.88810598218762300658	0.88810598218761237564	1.0631(-14)	6.4379(-06)	1.8900(-03)	6.4386(-06)	6.4381(-06)
0.96	1.11440179372400284780	1.11440179374552187898	2.1519(-11)	1.1717(-05)	2.7900(-03)	1.1718(-05)	1.1718(-05)

5. Discussion of Results

The analysis of the basic properties of the New Numerical Scheme (NNS) provides a theoretical foundation to justify its efficiency and reliability for solving Volterra integro-differential equations of the second kind. The order

and error constant are first examined by expanding the associated linear operators with a Taylor series. This allows the determination of the local truncation error, which directly measures the accuracy of the method. A higher order indicates better accuracy, while the error constant provides a bound on the size of the local error. By establishing these results through Corollaries 1 and 2, the study confirms that the proposed scheme attains a sufficiently high order with a controlled error constant, ensuring dependable approximations when applied to practical problems.

Consistency, zero-stability, and convergence are next established as interdependent properties. Consistency ensures that the truncation error approaches zero as the step size tends to zero, meaning the method approximates the exact differential operator correctly. Zero-stability, on the other hand, guarantees that small perturbations in initial conditions or rounding errors do not magnify uncontrollably, provided the roots of the characteristic equation lie within or on the unit disk. Since the NNS is both consistent and zero-stable, it satisfies the conditions for convergence. Therefore, the scheme not only produces accurate results locally but also ensures that errors remain bounded across successive iterations, reinforcing its theoretical soundness. Finally, the region of absolute stability was derived using the boundary locus method, which provides deeper insight into the robustness of the scheme. By applying the linear test equation, the stability function and corresponding polynomial are obtained. From the region plotted, it is A-stable.

The numerical examples demonstrate the efficiency and reliability of the New Numerical Scheme (NNS) compared to other established methods for solving Volterra integro-differential equations of the second kind. In Example 4.1, the scheme was applied to a test problem, and its results were compared with methods such as the Trapezoidal Method (TM), the Extended Trapezoidal Method (ETM), and the New Extended Trapezoidal Method (NETM). The comparisons across different step sizes revealed that the NNS consistently produced results that were much closer to the exact solution, thereby highlighting its superior accuracy and effectiveness over the existing schemes (See Tables 1 and 2).

In Example 2, the performance of the NNS was further assessed by comparing it with the Adams-Bashforth-Moulton method of order five (ABM5), the Two-Point Three-Step Block Method (2P3BVIDE), and the Boundary Trigonometrically Fitted Method (BTfM). The findings show that the NNS provided solutions that aligned almost perfectly with the exact solution, while the other methods showed larger deviations (see Table 3). This reinforces the robustness of the scheme and its ability to deliver accurate results across a wide range of step sizes, confirming its suitability for practical applications that require high precision.

The third test, Example 3, involved a nonlinear Volterra integro-differential equation, which typically presents greater numerical challenges. The NNS was compared with the Continuous Multistep Method (CMM), the Fitted Block Backward Differentiation Formula (FDTFBBDF), and multistep block methods (4SBM and 5SBM). The comparisons again confirmed the superiority of the NNS, as it produced solutions that were much closer to the exact solution than the competing methods. This demonstrates in Table 4 not only the high accuracy of the NNS but also its versatility in handling both linear and nonlinear problems effectively, making it a reliable and powerful numerical tool.

6. Conclusion

This study presented a New Numerical Scheme (NNS) for solving Volterra Integro-Differential Equations of the second kind, derived using a linear block algorithm. The method was analyzed for fundamental numerical properties, including order, error constant, consistency, zero-stability, convergence, and region of absolute stability. The theoretical analysis showed that the scheme possesses high order with a small error constant, is consistent and zero-stable, and therefore convergent. Furthermore, the region of absolute stability derived via the boundary locus method demonstrated that the scheme remains reliable even for stiff and oscillatory problems, with strong indications of A-stability.

To validate the theoretical findings, the scheme was applied to several linear and nonlinear Volterra integro-differential equations. Numerical experiments compared the NNS with existing methods such as the Trapezoidal Method (TM), Extended Trapezoidal Method (ETM), Adams-Bashforth-Moulton method (ABM5), and other block methods. In all test cases, the NNS consistently outperformed the existing schemes by producing results closer to the exact solutions, with negligible deviations across different step sizes. These results demonstrate the robustness, efficiency, and accuracy of the proposed method for both linear and nonlinear cases.

The proposed NNS proves to be a powerful and reliable approach for solving Volterra integro-differential equations of the second kind. Its strong theoretical foundation, confirmed by high-order accuracy, convergence, and a broad stability region, ensures dependable performance in various applications. The numerical results further confirm its superiority over existing methods, showing that it delivers highly accurate solutions with remarkable stability, even in nonlinear and stiff problem settings. Consequently, the NNS stands out as a valuable numerical tool for advancing the study and application of integral and integro-differential equations in scientific and engineering fields.

Authors' Contributions

This research article was organized in Collaboration with several authors. Conceptualization, Skwame and Donald; Methodology (derivation of the NNS), Analysis of the NNS Validation and numerical examples, Sabo.

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