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Original Paper

An Efficient Based Numerical Method for Accurate Integration of Volterra Integro-Differential Equations

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Abstract - This study presents a highly efficient New Numerical Method (NNM) for the accurate integration of second-kind Volterra Integro-Differential Equations (VIDEs). The New Numerical Method (NNM) is derived using a block algorithm based on a generalized linear multistep framework, enabling the construction of high-order schemes with multiple grid points. Analytical properties of the NNM, including order, error constants, consistency, zero-stability, convergence, and the region of absolute stability, are rigorously established to ensure reliability and robustness. Numerical simulations on representative VIDEs demonstrate the method's superior accuracy and stability compared to existing techniques such as Adams–Bashforth–Moulton predictor-corrector methods, general linear methods, trigonometrically fitted schemes, and Haar wavelet methods. The results indicate that the NNM achieves exact or near-exact solutions across various step sizes, highlighting its potential as a powerful computational tool for solving complex integro-differential equations in science and engineering applications.

Keywords - Volterra Integro-Differential Equations, New Numerical Method (NNM), Block Algorithm, High-Order Numerical Integration, Region of Absolute Stability.

1. Introduction

Integral equations are vital in science, engineering, and technology for modeling systems with cumulative effects, interdependencies, and boundary constraints. They enable the analysis of complex phenomena in physics, fluid and quantum mechanics, and mechanical and thermal sciences, including stress, strain, and heat conduction [1, 2]. Their compact mathematical representation enhances both analytical and computational understanding, underpinning key engineering methods like the Boundary Element Method and applications in electrical engineering, antenna design, circuit optimization, and computer science for image reconstruction and artificial intelligence [3-5]. Integral equations are commonly used to solve various problems in mathematical physics. A standard representation of such an equation is given by the form

$$\rho(\xi)=\vartheta(\xi)+\varphi\int_{\omega(\xi)}^{\varpi(\xi)}\mathrm{K}(\xi,\tau)\rho(\tau)d\tau \qquad (1)$$

Where φ is a constant parameter, $\mathrm{K}(\xi,\tau)$ is called the kernel of the integral equation, $\vartheta(\xi)$ is a function, and $\omega(\xi)$ and $\varpi(\xi)$ are the limits of integration, which can be constants, variables, or a combination of both. In

Equation (1), it is evident that the unknown function appears within the integral expression $\rho(\xi)$; the function to be determined typically appears both under the integral sign and in many cases, outside of it [6].

This study considers the numerical integration of Volterra Integro-Differential Equations (VIDEs) of the second kind of the form

$$\rho^{\lambda}(\xi)=\vartheta(\xi)+\varphi\int_0^{\xi}\mathrm{K}(\xi,\tau)\rho(\tau)d\tau \qquad (2)$$

The study of Volterra Integro-differential equations of the form (2) has gained significant attention due to their applications in various scientific and engineering fields. Traditional methods for solving VIEs, such as the direct computation method, Adomian decomposition method, variational iteration method, successive approximations method, and successive substitutions method, have some setbacks. However, the major setbacks of existing numerical methods for solving Volterra integral equations are computational inefficiencies when applied to higher-order Volterra integral equations and difficulties in the implementation of unrealistic series [1-3].

Research on second-kind Volterra Integro-Differential Equations (VIDEs) has gained significant interest because of their wide-ranging applications in science and engineering. Conventional methods for solving these equations, such as direct computation, Adomian decomposition, variational iteration, successive approximations, and successive substitutions, often encounter limitations, including high computational cost for higher-order problems and practical difficulties in applying series solutions [6, 7].

Recent developments in numerical methods for Volterra Integro-Differential Equations (VIDEs) have focused on improving accuracy, stability, and computational efficiency. Reference [8] refined the Extended Trapezoidal Method (ETM) by developing a PECE-mode algorithm that applies higher-order implicit formulations uniformly to both differential and integral components, addressing mismatched accuracy in earlier methods. Similarly, author [9] developed a fifth-order multistep block method based on a two-point, three-step Adams–Moulton framework with Boole's quadrature, enabling simultaneous computation of multiple solution points and demonstrating a wide stability region. The study by [10] introduced a continuous multistep approach using shifted Legendre polynomials with trapezoidal quadrature, while the author [11] applied a third-order General Linear Method combined with Simpson's and Lagrange quadrature, deriving coefficients that minimized principal error norms. These contributions collectively enhance the precision and robustness of numerical schemes for VIDEs.

Other approaches have addressed the behavior and computational costs inherent in traditional quadrature-based methods. The author [12] developed a Third-Derivative Trigonometrically Fitted Simpson's Method (TDTFBSM) with a block-by-block integration strategy, capable of simultaneously approximating multiple solution points. Building on this, study [13] presented a fifth-order trigonometrically fitted block method (BTFM) using a multistep collocation approach, ensuring zero-stability, consistency, and convergence.

The Adomian Decomposition Method (ADM), introduced by George Adomian in the 1970s–1990s, is a semi-analytical technique for solving linear and nonlinear differential equations without linearization or perturbation, using Adomian polynomials to decompose complex problems into rapidly convergent series [14]. ADM has been applied to a wide range of problems in physics, engineering, and stochastic systems [15, 16], though it can be sensitive to initial conditions and computationally intensive for higher-order terms [17, 18]. Complementary methods include the Variational Iteration Method (VIM), which employs Lagrange multipliers to construct correction functionals and achieves fast convergence for nonlinear equations [7, 19, 20], and the Direct Computation Method (DCM), which converts Volterra integral equations into algebraic systems for direct numerical solution [4, 21, 22]. Advances in hybrid techniques, including modified VIM with wavelets, DCM with orthogonal polynomials, and transform-based schemes, have improved convergence, accuracy, and applicability, highlighting the evolving landscape of semi-analytical and numerical methods for efficiently solving complex differential and integral equations [14, 22, 23].

The studies by reference [2, 24, 25] collectively advance numerical strategies for Volterra Integro-Differential Equations (VIDEs) by proposing high-accuracy and robust computational frameworks tailored to different problem structures. Reference [2] develops a versatile variational iteration–collocation method using shifted Chebyshev polynomials, demonstrating strong convergence and superior accuracy across higher-order VIDEs. Reference [24] focuses on singularly perturbed first-order VIDEs with integral boundary conditions, introducing a finite difference scheme that achieves uniform first-order convergence and effectively resolves boundary layer behavior where classical methods fail. Complementing these approaches, the author [25] applies a sixth-order Runge–Kutta technique, showing that high-order time-stepping significantly enhances precision and stability in solving general integro-differential models. Together, these works highlight ongoing progress in developing efficient, stable, and accurate numerical methods for diverse classes of VIDEs.

In order to overcome the setbacks in existing methods, this study presents a highly efficient New Numerical Method (NNM) for the accurate integration of second-kind Volterra Integro-Differential Equations (VIDEs). The new numerical method was derived using the linear block algorithm.

2. Materials and Methods

The New Numerical Method (NNM) was derived using the block algorithm with the help of Proposition 1.

2.1. Proposition 1

The general linear multistep method of the form

$$\sum_{j=0}^{1}\alpha_j\rho_{n+j}=h^{\mu}\sum_{j=0}^{1}\beta_j\vartheta_{n+j} \tag{3}$$

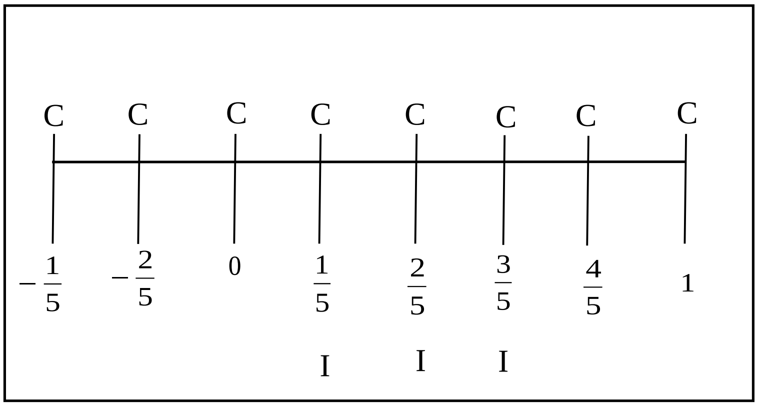
exists only an NNM from every single-step method. The block algorithm of the form

$$\rho_{n+\eta}=\sum_{j=0}^{2}\frac{(\eta h)^j}{j!}\rho_n^{(j)}+\sum_{j=0}^{1}\left(\Lambda_{j\eta}\vartheta_{n+j}\right),\quad \eta=-\frac{1}{5},-\frac{2}{5},0,\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},1 \tag{4}$$

is considered with its higher derivatives of the form,

$$\rho_{n+\eta}^{\sigma}=\sum_{j=0}^{2-\tau}\frac{(\eta h)^j}{j!}\rho_n^{(j+\sigma)}+\sum_{j=0}^{7}\left(\mathrm{X}_{j\sigma}\,\vartheta_{n+j}\right),\ \sigma=1_{\left(-\frac{1}{5},-\frac{2}{5},0,\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},1\right)},\ \sigma=2_{\left(-\frac{1}{5},-\frac{2}{5},0,\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},1\right)} \tag{5}$$

Where $\Lambda_{\eta j}=\Psi^{-1}\mathrm{Z}$, $\mathrm{X}_{\eta j\sigma}=\Psi^{-1}\mathrm{E}$ and



$$\Psi=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{\left(-\frac{1}{5}h\right)^1}{1!} & \frac{\left(-\frac{2}{5}h\right)^1}{1!} & \frac{(0)^1}{1!} & \frac{\left(\frac{1}{5}h\right)^1}{1!} & \frac{\left(\frac{2}{5}h\right)^1}{1!} & \frac{\left(\frac{3}{5}h\right)^1}{1!} & \frac{\left(\frac{4}{5}h\right)^1}{1!} & \frac{(h)^1}{1!} \\ \frac{\left(-\frac{1}{5}h\right)^2}{2!} & \frac{\left(-\frac{2}{5}h\right)^2}{2!} & \frac{(0)^2}{2!} & \frac{\left(\frac{1}{5}h\right)^2}{2!} & \frac{\left(\frac{2}{5}h\right)^2}{2!} & \frac{\left(\frac{3}{5}h\right)^2}{2!} & \frac{\left(\frac{4}{5}h\right)^2}{2!} & \frac{(h)^2}{2!} \\ \frac{\left(-\frac{1}{5}h\right)^3}{3!} & \frac{\left(-\frac{2}{5}h\right)^3}{3!} & \frac{(0)^3}{3!} & \frac{\left(\frac{1}{5}h\right)^3}{3!} & \frac{\left(\frac{2}{5}h\right)^3}{3!} & \frac{\left(\frac{3}{5}h\right)^3}{3!} & \frac{\left(\frac{4}{5}h\right)^3}{3!} & \frac{(h)^3}{3!} \\ \frac{\left(-\frac{1}{5}h\right)^4}{4!} & \frac{\left(-\frac{2}{5}h\right)^4}{4!} & \frac{(0)^4}{4!} & \frac{\left(\frac{1}{5}h\right)^4}{4!} & \frac{\left(\frac{2}{5}h\right)^4}{4!} & \frac{\left(\frac{3}{5}h\right)^4}{4!} & \frac{\left(\frac{4}{5}h\right)^4}{4!} & \frac{(h)^4}{4!} \\ \frac{\left(-\frac{1}{5}h\right)^5}{5!} & \frac{\left(-\frac{2}{5}h\right)^5}{5!} & \frac{(0)^5}{5!} & \frac{\left(\frac{1}{5}h\right)^5}{5!} & \frac{\left(\frac{2}{5}h\right)^5}{5!} & \frac{\left(\frac{3}{5}h\right)^5}{5!} & \frac{\left(\frac{4}{5}h\right)^5}{5!} & \frac{(h)^5}{5!} \\ \frac{\left(-\frac{1}{5}h\right)^6}{6!} & \frac{\left(-\frac{2}{5}h\right)^6}{6!} & \frac{(0)^6}{6!} & \frac{\left(\frac{1}{5}h\right)^6}{6!} & \frac{\left(\frac{2}{5}h\right)^6}{6!} & \frac{\left(\frac{3}{5}h\right)^6}{6!} & \frac{\left(\frac{4}{5}h\right)^6}{6!} & \frac{(h)^6}{6!} \\ \frac{\left(-\frac{1}{5}h\right)^7}{7!} & \frac{\left(-\frac{2}{5}h\right)^7}{7!} & \frac{(0)^7}{7!} & \frac{\left(\frac{1}{5}h\right)^7}{7!} & \frac{\left(\frac{2}{5}h\right)^7}{7!} & \frac{\left(\frac{3}{5}h\right)^7}{7!} & \frac{\left(\frac{4}{5}h\right)^7}{7!} & \frac{(h)^7}{7!} \end{pmatrix},\ Z=\begin{pmatrix} \frac{(\xi h)^3}{3!} \\ \frac{(\xi h)^4}{4!} \\ \frac{(\xi h)^5}{5!} \\ \frac{(\xi h)^6}{6!} \\ \frac{(\xi h)^7}{7!} \\ \frac{(\xi h)^8}{8!} \\ \frac{(\xi h)^9}{9!} \\ \frac{(\xi h)^{10}}{10!} \end{pmatrix},\ E=\begin{pmatrix} \frac{(\xi h)^{3-\sigma}}{(3-\sigma)!} \\ \frac{(\xi h)^{4-\sigma}}{(4-\sigma)!} \\ \frac{(\xi h)^{5-\sigma}}{(5-\sigma)!} \\ \frac{(\xi h)^{6-\sigma}}{(6-\sigma)!} \\ \frac{(\xi h)^{7-\sigma}}{(7-\sigma)!} \\ \frac{(\xi h)^{8-\sigma}}{(8-\sigma)!} \\ \frac{(\xi h)^{9-\sigma}}{(9-\sigma)!} \\ \frac{(\xi h)^{10-\sigma}}{(10-\sigma)!} \end{pmatrix}$$

2.2. Proof

In order to obtain a new third-derivative numerical scheme with six grid points, equations (4) and (5) are solved sequentially to derive a polynomial of the form

$$\rho(\xi_n+\eta h)=\alpha_{\frac{1}{5}}\rho_{n+\frac{1}{5}}+\rho_{\frac{2}{5}}\rho_{n+\frac{2}{5}}+\rho_{\frac{3}{5}}\rho_{n+\frac{3}{5}}+h^3\left(\beta_{-\frac{1}{5}}\vartheta_{n-\frac{1}{5}}+\beta_{-\frac{2}{5}}\vartheta_{n-\frac{2}{5}}+\beta_0\vartheta_n+\beta_{\frac{1}{5}}\vartheta_{n+\frac{1}{5}}+\beta_{\frac{2}{5}}\vartheta_{n+\frac{2}{5}}+\beta_{\frac{3}{5}}\vartheta_{n+\frac{3}{5}}+\beta_{\frac{4}{5}}\vartheta_{n+\frac{4}{5}}+\beta_1\vartheta_{n+1}\right) \quad (6)$$

Where $\eta=\xi_n+\xi h$ in the polynomial (6) and the continuous form of NNM are

$$\left.\begin{aligned}
&\alpha_{\frac{1}{5}} = 3-\frac{25}{2}\eta+\frac{25}{2}\eta^{2},\ \alpha_{\frac{2}{5}} = -3+20\eta-25\eta^{2},\ \alpha_{\frac{3}{5}} = 1-\frac{15}{2}\eta+\frac{25}{2}\eta^{2},\\
&\beta_{-\frac{1}{5}} = -\frac{281}{7560000}-\frac{1441}{45360000}\eta+\frac{211}{90720}\eta^{2}-\frac{5}{72}\eta^{4}+\frac{107}{432}\eta^{5}-\frac{55}{216}\eta^{6}-\frac{1375}{6048}\eta^{7}+\frac{34375}{48384}\eta^{8}-\frac{40625}{72576}\eta^{9}+\frac{3125}{20736}\eta^{10}\\
&\beta_{-\frac{2}{5}} = \frac{31}{7560000}-\frac{133}{22680000}\eta-\frac{645}{3628800}\eta^{2}+\frac{5}{1008}\eta^{4}-\frac{11}{864}\eta^{5}-\frac{35}{3456}\eta^{6}+\frac{125}{1512}\eta^{7}-\frac{3125}{24192}\eta^{8}+\frac{3125}{36288}\eta^{9}-\frac{3125}{145152}\eta^{10}\\
&\beta_{0} = \frac{73}{840000}+\frac{617}{210000}\eta-\frac{2017}{48384}\eta^{2}+\frac{1}{6}\eta^{4}-\frac{47}{288}\eta^{5}-\frac{7}{16}\eta^{6}+\frac{385}{384}\eta^{7}-\frac{625}{384}\eta^{8}+\frac{3125}{2016}\eta^{9}-\frac{3125}{6912}\eta^{10}\\
&\beta_{\frac{1}{5}} = -\frac{30647}{7560000}+\frac{371773}{9072000}\eta-\frac{52907}{453600}\eta^{2}+\frac{25}{72}\eta^{4}+\frac{65}{432}\eta^{5}-\frac{625}{432}\eta^{6}+\frac{3875}{6048}\eta^{7}+\frac{96875}{48384}\eta^{8}-\frac{171875}{72576}\eta^{9}+\frac{15625}{20736}\eta^{10}\\
&\beta_{\frac{2}{5}} = -\frac{30647}{7560000}+\frac{19121}{648000}\eta-\frac{149993}{3628800}\eta^{2}-\frac{25}{144}\eta^{4}+\frac{85}{864}\eta^{5}+\frac{3575}{3456}\eta^{6}-\frac{1375}{1512}\eta^{7}-\frac{34375}{24192}\eta^{8}+\frac{78125}{36288}\eta^{9}-\frac{15625}{20736}\eta^{10}\\
&\beta_{\frac{3}{5}} = \frac{73}{840000}-\frac{1031}{5040000}\eta-\frac{2}{675}\eta^{2}+\frac{5}{72}\eta^{4}-\frac{1}{16}\eta^{5}-\frac{5}{12}\eta^{6}+\frac{125}{224}\eta^{7}+\frac{3125}{5376}\eta^{8}-\frac{3125}{2688}\eta^{9}+\frac{3125}{6912}\eta^{10}\\
&\beta_{\frac{4}{5}} = -\frac{281}{7560000}+\frac{1889}{11340000}\eta+\frac{1943}{3628800}\eta^{2}-\frac{5}{288}\eta^{4}+\frac{1}{54}\eta^{5}+\frac{355}{3456}\eta^{6}-\frac{125}{756}\eta^{7}-\frac{3125}{24192}\eta^{8}+\frac{3125}{9072}\eta^{9}-\frac{3125}{20736}\eta^{10}\\
&\beta_{1} = \frac{31}{7560000}-\frac{817}{45360000}\eta-\frac{1}{16200}\eta^{2}+\frac{1}{504}\eta^{4}-\frac{1}{432}\eta^{5}-\frac{5}{432}\eta^{6}+\frac{125}{6048}\eta^{7}+\frac{625}{48384}\eta^{8}-\frac{3125}{72576}\eta^{9}+\frac{3125}{145152}\eta^{10}
\end{aligned}\right\}\qquad (7)$$

Expand equation (4) to obtain the generalized NNM as

$$\left.\begin{aligned}
&\rho_{n-\frac{1}{5}} = \rho_n-\frac{1}{5}h\rho'_n+\frac{\left(-\frac{1}{5}h\right)^{2}}{2!}\rho''_n+h^{3}\left(\Lambda_{10}\vartheta_{n-\frac{1}{5}}+\Lambda_{11}\vartheta_{n-\frac{2}{5}}+\Lambda_{12}\vartheta_{n}+\Lambda_{13}\vartheta_{n+\frac{1}{5}}+\Lambda_{14}\vartheta_{n+\frac{2}{5}}+\Lambda_{15}\vartheta_{n+\frac{3}{5}}+\Lambda_{16}\vartheta_{n+\frac{4}{5}}+\Lambda_{17}\vartheta_{n+1}\right)\\
&\rho_{n-\frac{2}{5}} = \rho_n-\frac{2}{5}h\rho'_n+\frac{\left(-\frac{2}{5}h\right)^{2}}{2!}\rho''_n+h^{3}\left(\Lambda_{20}\vartheta_{n-\frac{1}{5}}+\Lambda_{21}\vartheta_{n-\frac{2}{5}}+\Lambda_{22}\vartheta_{n}+\Lambda_{23}\vartheta_{n+\frac{1}{5}}+\Lambda_{24}\vartheta_{n+\frac{2}{5}}+\Lambda_{25}\vartheta_{n+\frac{3}{5}}+\Lambda_{26}\vartheta_{n+\frac{4}{5}}+\Lambda_{27}\vartheta_{n+1}\right)\\
&\rho_{n+\frac{1}{5}} = \rho_n+\frac{1}{5}h\rho'_n+\frac{\left(\frac{1}{5}h\right)^{2}}{2!}\rho''_n+h^{3}\left(\Lambda_{30}\vartheta_{n-\frac{1}{5}}+\Lambda_{31}\vartheta_{n-\frac{2}{5}}+\Lambda_{32}\vartheta_{n}+\Lambda_{33}\vartheta_{n+\frac{1}{5}}+\Lambda_{34}\vartheta_{n+\frac{2}{5}}+\Lambda_{35}\vartheta_{n+\frac{3}{5}}+\Lambda_{36}\vartheta_{n+\frac{4}{5}}+\Lambda_{37}\vartheta_{n+1}\right)\\
&\rho_{n+\frac{2}{5}} = \rho_n+\frac{2}{5}h\rho'_n+\frac{\left(\frac{2}{5}h\right)^{2}}{2!}\rho''_n+h^{3}\left(\Lambda_{40}\vartheta_{n-\frac{1}{5}}+\Lambda_{41}\vartheta_{n-\frac{2}{5}}+\Lambda_{42}\vartheta_{n}+\Lambda_{43}\vartheta_{n+\frac{1}{5}}+\Lambda_{44}\vartheta_{n+\frac{2}{5}}+\Lambda_{45}\vartheta_{n+\frac{3}{5}}+\Lambda_{46}\vartheta_{n+\frac{4}{5}}+\Lambda_{47}\vartheta_{n+1}\right)\\
&\rho_{n+\frac{3}{5}} = \rho_n+\frac{3}{5}h\rho'_n+\frac{\left(\frac{3}{5}h\right)^{2}}{2!}\rho''_n+h^{3}\left(\Lambda_{50}\vartheta_{n-\frac{1}{5}}+\Lambda_{51}\vartheta_{n-\frac{2}{5}}+\Lambda_{52}\vartheta_{n}+\Lambda_{53}\vartheta_{n+\frac{1}{5}}+\Lambda_{54}\vartheta_{n+\frac{2}{5}}+\Lambda_{55}\vartheta_{n+\frac{3}{5}}+\Lambda_{56}\vartheta_{n+\frac{4}{5}}+\Lambda_{57}\vartheta_{n+1}\right)\\
&\rho_{n+\frac{4}{5}} = \rho_n+\frac{4}{5}h\rho'_n+\frac{\left(\frac{4}{5}h\right)^{2}}{2!}\rho''_n+h^{3}\left(\Lambda_{60}\vartheta_{n-\frac{1}{5}}+\Lambda_{61}\vartheta_{n-\frac{2}{5}}+\Lambda_{62}\vartheta_{n}+\Lambda_{63}\vartheta_{n+\frac{1}{5}}+\Lambda_{64}\vartheta_{n+\frac{2}{5}}+\Lambda_{65}\vartheta_{n+\frac{3}{5}}+\Lambda_{66}\vartheta_{n+\frac{4}{5}}+\Lambda_{67}\vartheta_{n+1}\right)\\
&\rho_{n+\frac{1}{5}} = \rho_n+h\rho'_n+\frac{(h)^{2}}{2!}\rho''_n+h^{3}\left(\Lambda_{70}\vartheta_{n-\frac{1}{5}}+\Lambda_{71}\vartheta_{n-\frac{2}{5}}+\Lambda_{72}\vartheta_{n}+\Lambda_{73}\vartheta_{n+\frac{1}{5}}+\Lambda_{74}\vartheta_{n+\frac{2}{5}}+\Lambda_{75}\vartheta_{n+\frac{3}{5}}+\Lambda_{76}\vartheta_{n+\frac{4}{5}}+\Lambda_{77}\vartheta_{n+1}\right)
\end{aligned}\right\}\qquad (8)$$

Its higher derivatives are below as

$$\left.\begin{aligned}
\rho'_{n-\frac{1}{5}} &= \rho'_n - \frac{1}{5}h\rho''_n + h^2 f\left(X_{110}\vartheta_{n-\frac{1}{5}} + X_{111}\vartheta_{n-\frac{2}{5}} + X_{112}\vartheta_n + X_{113}\vartheta_{n+\frac{1}{5}} + X_{114}\vartheta_{n+\frac{2}{5}} + X_{115}\vartheta_{n+\frac{3}{5}} + X_{116}\vartheta_{n+\frac{4}{5}} + X_{117}\vartheta_{n+1}\right)\\
\rho'_{n-\frac{2}{5}} &= \rho'_n - \frac{2}{5}h\rho''_n + h^2\left(X_{120}\vartheta_{n-\frac{1}{5}} + X_{121}\vartheta_{n-\frac{2}{5}} + X_{122}\vartheta_n + X_{123}\vartheta_{n+\frac{1}{5}} + X_{124}\vartheta_{n+\frac{2}{5}} + X_{125}\vartheta_{n+\frac{3}{5}} + X_{126}\vartheta_{n+\frac{4}{5}} + X_{127}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{1}{5}} &= \rho'_n + \frac{1}{5}h\rho''_n + h^2\left(X_{130}\vartheta_{n-\frac{1}{5}} + X_{131}\vartheta_{n-\frac{2}{5}} + X_{132}\vartheta_n + X_{133}\vartheta_{n+\frac{1}{5}} + X_{134}\vartheta_{n+\frac{2}{5}} + X_{135}\vartheta_{n+\frac{3}{5}} + X_{136}\vartheta_{n+\frac{4}{5}} + X_{137}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{2}{5}} &= \rho'_n + \frac{2}{5}h\rho''_n + h^2\left(X_{140}\vartheta_{n-\frac{1}{5}} + X_{141}\vartheta_{n-\frac{2}{5}} + X_{142}\vartheta_n + X_{143}\vartheta_{n+\frac{1}{5}} + X_{144}\vartheta_{n+\frac{2}{5}} + X_{145}\vartheta_{n+\frac{3}{5}} + X_{146}\vartheta_{n+\frac{4}{5}} + X_{147}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{3}{5}} &= \rho'_n + \frac{3}{5}h\rho''_n + h^2\left(X_{150}\vartheta_{n-\frac{1}{5}} + X_{151}\vartheta_{n-\frac{2}{5}} + X_{152}\vartheta_n + X_{153}\vartheta_{n+\frac{1}{5}} + X_{154}\vartheta_{n+\frac{2}{5}} + X_{155}\vartheta_{n+\frac{3}{5}} + X_{156}\vartheta_{n+\frac{4}{5}} + X_{157}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{4}{5}} &= \rho'_n + \frac{4}{5}h\rho''_n + h^2\left(X_{160}\vartheta_{n-\frac{1}{5}} + X_{161}\vartheta_{n-\frac{2}{5}} + X_{162}\vartheta_n + X_{163}\vartheta_{n+\frac{1}{5}} + X_{164}\vartheta_{n+\frac{2}{5}} + X_{165}\vartheta_{n+\frac{3}{5}} + X_{166}\vartheta_{n+\frac{4}{5}} + X_{167}\vartheta_{n+1}\right)\\
\rho'_{n+1} &= \rho'_n + h\rho''_n + h^2\left(X_{170}\vartheta_{n-\frac{1}{5}} + X_{171}\vartheta_{n-\frac{2}{5}} + X_{172}\vartheta_n + X_{173}\vartheta_{n+\frac{1}{5}} + X_{174}\vartheta_{n+\frac{2}{5}} + X_{175}\vartheta_{n+\frac{3}{5}} + X_{176}\vartheta_{n+\frac{4}{5}} + X_{177}\vartheta_{n+1}\right)
\end{aligned}\right\} \qquad (9)$$

$$\left.\begin{aligned}
\rho''_{n-\frac{1}{5}} &= \rho''_n + h\left(X_{210}\vartheta_{n-\frac{1}{5}} + X_{211}\vartheta_{n-\frac{2}{5}} + X_{212}\vartheta_n + X_{213}\vartheta_{n+\frac{1}{5}} + X_{214}\vartheta_{n+\frac{2}{5}} + X_{215}\vartheta_{n+\frac{3}{5}} + X_{216}\vartheta_{n+\frac{4}{5}} + X_{217}\vartheta_{n+1}\right)\\
\rho''_{n-\frac{2}{5}} &= \rho''_n + h\left(X_{220}\vartheta_{n-\frac{1}{5}} + X_{221}\vartheta_{n-\frac{2}{5}} + X_{222}\vartheta_n + X_{223}\vartheta_{n+\frac{1}{5}} + X_{224}\vartheta_{n+\frac{2}{5}} + X_{225}\vartheta_{n+\frac{3}{5}} + X_{226}\vartheta_{n+\frac{4}{5}} + X_{227}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{1}{5}} &= \rho''_n + h\left(X_{230}\vartheta_{n-\frac{1}{5}} + X_{231}\vartheta_{n-\frac{2}{5}} + X_{232}\vartheta_n + X_{233}\vartheta_{n+\frac{1}{5}} + X_{234}\vartheta_{n+\frac{2}{5}} + X_{235}\vartheta_{n+\frac{3}{5}} + X_{236}\vartheta_{n+\frac{4}{5}} + X_{237}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{2}{5}} &= \rho''_n + h\left(X_{240}\vartheta_{n-\frac{1}{5}} + X_{241}\vartheta_{n-\frac{2}{5}} + X_{242}\vartheta_n + X_{243}\vartheta_{n+\frac{1}{5}} + X_{244}\vartheta_{n+\frac{2}{5}} + X_{245}\vartheta_{n+\frac{3}{5}} + X_{246}\vartheta_{n+\frac{4}{5}} + X_{247}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{3}{5}} &= \rho''_n + h\left(X_{250}\vartheta_{n-\frac{1}{5}} + X_{251}\vartheta_{n-\frac{2}{5}} + X_{252}\vartheta_n + X_{253}\vartheta_{n+\frac{1}{5}} + X_{254}\vartheta_{n+\frac{2}{5}} + X_{255}\vartheta_{n+\frac{3}{5}} + X_{256}\vartheta_{n+\frac{4}{5}} + X_{257}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{4}{5}} &= \rho''_n + h\left(X_{260}\vartheta_{n-\frac{1}{5}} + X_{261}\vartheta_{n-\frac{2}{5}} + X_{262}\vartheta_n + X_{263}\vartheta_{n+\frac{1}{5}} + X_{264}\vartheta_{n+\frac{2}{5}} + X_{265}\vartheta_{n+\frac{3}{5}} + X_{266}\vartheta_{n+\frac{4}{5}} + X_{267}\vartheta_{n+1}\right)\\
\rho''_{n+1} &= \rho''_n + h\left(X_{270}\vartheta_{n-\frac{1}{5}} + X_{271}\vartheta_{n-\frac{2}{5}} + X_{272}\vartheta_n + X_{273}\vartheta_{n+\frac{1}{5}} + X_{274}\vartheta_{n+\frac{2}{5}} + X_{275}\vartheta_{n+\frac{3}{5}} + X_{276}\vartheta_{n+\frac{4}{5}} + X_{277}\vartheta_{n+1}\right)
\end{aligned}\right\} \qquad (10)$$

Simplify $\Lambda_{\eta j} = \Psi^{-1}Z$ in equation (4) to the unknown coefficients of Λ in equation (8) as

$$\begin{pmatrix}\Lambda_{10}\\ \Lambda_{11}\\ \Lambda_{12}\\ \Lambda_{13}\\ \Lambda_{14}\\ \Lambda_{15}\\ \Lambda_{16}\\ \Lambda_{17}\end{pmatrix} = \begin{pmatrix}-\frac{6533}{32400000}\\ \frac{1501}{151200000}\\ -\frac{210967}{151200000}\\ \frac{1249}{3024000}\\ -\frac{21431}{90720000}\\ \frac{7517}{75600000}\\ -\frac{143}{5600000}\\ \frac{673}{226800000}\end{pmatrix}, \begin{pmatrix}\Lambda_{20}\\ \Lambda_{21}\\ \Lambda_{22}\\ \Lambda_{23}\\ \Lambda_{24}\\ \Lambda_{25}\\ \Lambda_{26}\\ \Lambda_{27}\end{pmatrix} = \begin{pmatrix}-\frac{2573}{590625}\\ -\frac{103}{3543750}\\ -\frac{4594}{590625}\\ \frac{169}{70875}\\ -\frac{307}{236250}\\ \frac{313}{590625}\\ -\frac{236}{1771875}\\ \frac{1}{65625}\end{pmatrix}, \begin{pmatrix}\Lambda_{30}\\ \Lambda_{31}\\ \Lambda_{32}\\ \Lambda_{33}\\ \Lambda_{34}\\ \Lambda_{35}\\ \Lambda_{36}\\ \Lambda_{37}\end{pmatrix} = \begin{pmatrix}-\frac{3743}{75600000}\\ \frac{1807}{453600000}\\ \frac{150137}{151200000}\\ \frac{3391}{6480000}\\ -\frac{635}{3360000}\\ \frac{5483}{75600000}\\ -\frac{7967}{453600000}\\ \frac{149}{75600000}\end{pmatrix}, \begin{pmatrix}\Lambda_{40}\\ \Lambda_{41}\\ \Lambda_{42}\\ \Lambda_{43}\\ \Lambda_{44}\\ \Lambda_{45}\\ \Lambda_{46}\\ \Lambda_{47}\end{pmatrix} = \begin{pmatrix}-\frac{571}{1771875}\\ \frac{31}{1181250}\\ -\frac{3194}{590625}\\ \frac{83}{13125}\\ -\frac{23}{20250}\\ \frac{277}{590625}\\ -\frac{68}{590625}\\ \frac{23}{1771875}\end{pmatrix}, \begin{pmatrix}\Lambda_{50}\\ \Lambda_{51}\\ \Lambda_{52}\\ \Lambda_{53}\\ \Lambda_{54}\\ \Lambda_{55}\\ \Lambda_{56}\\ \Lambda_{57}\end{pmatrix} = \begin{pmatrix}-\frac{2187}{2800000}\\ \frac{351}{5600000}\\ \frac{73683}{5600000}\\ \frac{2403}{112000}\\ \frac{1377}{1120000}\\ \frac{441}{400000}\\ -\frac{1431}{5600000}\\ \frac{81}{2800000}\end{pmatrix}, \begin{pmatrix}\Lambda_{60}\\ \Lambda_{61}\\ \Lambda_{62}\\ \Lambda_{63}\\ \Lambda_{64}\\ \Lambda_{65}\\ \Lambda_{66}\\ \Lambda_{67}\end{pmatrix} = \begin{pmatrix}-\frac{23}{21875}\\ \frac{208}{1771875}\\ \frac{14408}{590625}\\ \frac{3232}{70875}\\ \frac{1328}{118125}\\ \frac{3424}{590625}\\ -\frac{104}{253125}\\ \frac{32}{590625}\end{pmatrix}, \begin{pmatrix}\Lambda_{70}\\ \Lambda_{71}\\ \Lambda_{72}\\ \Lambda_{73}\\ \Lambda_{74}\\ \Lambda_{75}\\ \Lambda_{76}\\ \Lambda_{77}\end{pmatrix} = \begin{pmatrix}-\frac{169}{72576}\\ \frac{1}{5376}\\ \frac{1885}{48384}\\ \frac{1915}{24192}\\ \frac{4105}{145152}\\ \frac{463}{24192}\\ \frac{151}{48384}\\ \frac{11}{72576}\end{pmatrix}$$

Similarly, simplify $X_{\eta j\sigma} = \Psi^{-1}E$, equation (5) to obtain the unknown coefficients of the higher derivative X in equations (9) and (10) as

$$\begin{pmatrix} X_{110} \\ X_{111} \\ X_{112} \\ X_{113} \\ X_{114} \\ X_{115} \\ X_{116} \\ X_{117} \end{pmatrix} = \begin{pmatrix} \frac{7297}{1620000} \\ \frac{8563}{45360000} \\ \frac{302429}{15120000} \\ -\frac{32233}{4536000} \\ \frac{37631}{9072000} \\ -\frac{208}{118125} \\ \frac{20609}{45360000} \\ -\frac{1201}{22680000} \end{pmatrix}, \begin{pmatrix} X_{120} \\ X_{121} \\ X_{122} \\ X_{123} \\ X_{124} \\ X_{125} \\ X_{126} \\ X_{127} \end{pmatrix} = \begin{pmatrix} \frac{16159}{354375} \\ \frac{119}{50625} \\ \frac{8467}{236250} \\ -\frac{323}{70875} \\ \frac{2}{2835} \\ \frac{19}{118125} \\ -\frac{83}{708750} \\ \frac{1}{50625} \end{pmatrix}, \begin{pmatrix} X_{130} \\ X_{131} \\ X_{132} \\ X_{133} \\ X_{134} \\ X_{135} \\ X_{136} \\ X_{137} \end{pmatrix} = \begin{pmatrix} -\frac{17483}{22680000} \\ \frac{409}{6480000} \\ \frac{197611}{15120000} \\ \frac{3233}{324000} \\ -\frac{29843}{9072000} \\ \frac{9127}{7560000} \\ -\frac{13169}{45360000} \\ \frac{23}{708750} \end{pmatrix}, \begin{pmatrix} X_{140} \\ X_{141} \\ X_{142} \\ X_{143} \\ X_{144} \\ X_{145} \\ X_{146} \\ X_{147} \end{pmatrix} = \begin{pmatrix} -\frac{649}{354375} \\ \frac{52}{354375} \\ -\frac{7193}{236250} \\ \frac{3677}{70875} \\ -\frac{23}{10125} \\ \frac{251}{118125} \\ -\frac{397}{708750} \\ \frac{23}{354375} \end{pmatrix}, \begin{pmatrix} X_{150} \\ X_{151} \\ X_{152} \\ X_{153} \\ X_{154} \\ X_{155} \\ X_{156} \\ X_{157} \end{pmatrix} = \begin{pmatrix} -\frac{201}{70000} \\ \frac{129}{560000} \\ \frac{26619}{560000} \\ \frac{5493}{56000} \\ \frac{168}{22400} \\ \frac{147}{20000} \\ -\frac{627}{560000} \\ \frac{33}{280000} \end{pmatrix}, \begin{pmatrix} X_{160} \\ X_{161} \\ X_{162} \\ X_{163} \\ X_{164} \\ X_{165} \\ X_{166} \\ X_{167} \end{pmatrix} = \begin{pmatrix} -\frac{1328}{354375} \\ \frac{104}{354375} \\ \frac{7568}{118125} \\ \frac{10288}{70875} \\ \frac{4712}{70875} \\ \frac{5392}{118125} \\ \frac{104}{50625} \\ \frac{16}{354375} \end{pmatrix}, \begin{pmatrix} X_{170} \\ X_{171} \\ X_{172} \\ X_{173} \\ X_{174} \\ X_{175} \\ X_{176} \\ X_{177} \end{pmatrix} = \begin{pmatrix} -\frac{205}{36288} \\ \frac{5}{10368} \\ \frac{2039}{24192} \\ \frac{1675}{9072} \\ \frac{8125}{72576} \\ \frac{965}{12096} \\ \frac{3035}{72576} \\ \frac{7}{2592} \end{pmatrix},$$

$$\begin{pmatrix} X_{210} \\ X_{211} \\ X_{241} \\ X_{213} \\ X_{214} \\ X_{215} \\ X_{216} \\ X_{217} \end{pmatrix} = \begin{pmatrix} -\frac{5311}{67200} \\ \frac{55}{24192} \\ -\frac{11261}{67200} \\ \frac{44797}{604800} \\ -\frac{2987}{67200} \\ \frac{1283}{67200} \\ -\frac{2999}{604800} \\ \frac{13}{22400} \end{pmatrix}, \begin{pmatrix} X_{220} \\ X_{221} \\ X_{222} \\ X_{223} \\ X_{224} \\ X_{225} \\ X_{226} \\ X_{227} \end{pmatrix} = \begin{pmatrix} -\frac{1466}{4725} \\ -\frac{41}{700} \\ \frac{71}{2100} \\ -\frac{68}{525} \\ \frac{1927}{18900} \\ -\frac{56}{525} \\ \frac{29}{2100} \\ \frac{8}{4725} \end{pmatrix}, \begin{pmatrix} X_{230} \\ X_{231} \\ X_{232} \\ X_{233} \\ X_{234} \\ X_{235} \\ X_{236} \\ X_{237} \end{pmatrix} = \begin{pmatrix} -\frac{4183}{604800} \\ \frac{13}{22400} \\ \frac{6403}{67200} \\ \frac{9077}{67200} \\ -\frac{20227}{604800} \\ \frac{803}{67200} \\ -\frac{191}{67200} \\ \frac{191}{604800} \end{pmatrix}, \begin{pmatrix} X_{240} \\ X_{241} \\ X_{242} \\ X_{243} \\ X_{244} \\ X_{245} \\ X_{246} \\ X_{247} \end{pmatrix} = \begin{pmatrix} -\frac{2}{525} \\ \frac{1}{3780} \\ \frac{167}{2100} \\ \frac{1172}{4725} \\ \frac{167}{2100} \\ -\frac{2}{525} \\ \frac{1}{3780} \\ 0 \end{pmatrix}, \begin{pmatrix} X_{250} \\ X_{251} \\ X_{252} \\ X_{253} \\ X_{254} \\ X_{255} \\ X_{256} \\ X_{257} \end{pmatrix} = \begin{pmatrix} -\frac{149}{22400} \\ \frac{13}{22400} \\ \frac{2049}{22400} \\ \frac{4807}{22400} \\ \frac{4807}{22400} \\ \frac{2049}{22400} \\ -\frac{149}{22400} \\ \frac{13}{22400} \end{pmatrix}, \begin{pmatrix} X_{260} \\ X_{261} \\ X_{262} \\ X_{263} \\ X_{264} \\ X_{265} \\ X_{266} \\ X_{267} \end{pmatrix} = \begin{pmatrix} -\frac{8}{4725} \\ 0 \\ \frac{38}{525} \\ \frac{138}{525} \\ \frac{664}{4725} \\ \frac{136}{525} \\ \frac{38}{525} \\ -\frac{8}{4725} \end{pmatrix}, \begin{pmatrix} X_{270} \\ X_{271} \\ X_{272} \\ X_{273} \\ X_{274} \\ X_{275} \\ X_{276} \\ X_{277} \end{pmatrix} = \begin{pmatrix} -\frac{55}{2688} \\ \frac{55}{24192} \\ \frac{379}{2688} \\ \frac{2725}{24192} \\ \frac{925}{2688} \\ \frac{155}{2688} \\ \frac{7345}{24192} \\ \frac{53}{896} \end{pmatrix}$$

3. Analysis of New Numerical Method (NNM)

These analyses of the basic properties of the New Numerical Method (NNM) are rigorously studied according to reference [26, 27]. These analyses are order and error constants, consistency, Zero-stability, Convergent, and region of absolute stability.

3.1. Order and Error Constant of LBA Scheme

Using Corollary 1 and Corollary 2 to obtain the order and error constant of NNM.

3.1.1. Corollary 3.1

The linear operator $L[\rho(\xi_n);h]$ associated with the local truncation error of the NNM in equation (8) and its higher derivatives in equations (9) and (10) is given by

$$C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{11}), C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{10}), C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{09}).$$

3.1.2. Proof

The linear difference operators associated with equations (8) through (10) are given by

$$
\left.
\begin{aligned}
L[\rho(\xi_n);h] &= \rho_{n-\frac{1}{5}} - \rho_n + \frac{1}{5}h\rho'_n - \frac{\left(-\frac{1}{5}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{10}\vartheta_{n-\frac{1}{5}} + \Lambda_{11}\vartheta_{n-\frac{2}{5}} + \Lambda_{12}\vartheta_n + \Lambda_{13}\vartheta_{n+\frac{1}{5}} + \Lambda_{14}\vartheta_{n+\frac{2}{5}} + \Lambda_{15}\vartheta_{n+\frac{3}{5}} + \Lambda_{16}\vartheta_{n+\frac{4}{5}} + \Lambda_{17}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n-\frac{2}{5}} - \rho_n + \frac{2}{5}h\rho'_n - \frac{\left(-\frac{2}{5}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{20}\vartheta_{n-\frac{1}{5}} + \Lambda_{21}\vartheta_{n-\frac{2}{5}} + \Lambda_{22}\vartheta_n + \Lambda_{23}\vartheta_{n+\frac{1}{5}} + \Lambda_{24}\vartheta_{n+\frac{2}{5}} + \Lambda_{25}\vartheta_{n+\frac{3}{5}} + \Lambda_{26}\vartheta_{n+\frac{4}{5}} + \Lambda_{27}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+\frac{1}{5}} - \rho_n - \frac{1}{5}h\rho'_n - \frac{\left(\frac{1}{5}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{30}\vartheta_{n-\frac{1}{5}} + \Lambda_{31}\vartheta_{n-\frac{2}{5}} + \Lambda_{32}\vartheta_n + \Lambda_{33}\vartheta_{n+\frac{1}{5}} + \Lambda_{34}\vartheta_{n+\frac{2}{5}} + \Lambda_{35}\vartheta_{n+\frac{3}{5}} + \Lambda_{36}\vartheta_{n+\frac{4}{5}} + \Lambda_{37}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+\frac{2}{5}} - \rho_n - \frac{2}{5}h\rho'_n - \frac{\left(\frac{2}{5}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{40}\vartheta_{n-\frac{1}{5}} + \Lambda_{41}\vartheta_{n-\frac{2}{5}} + \Lambda_{42}\vartheta_n + \Lambda_{43}\vartheta_{n+\frac{1}{5}} + \Lambda_{44}\vartheta_{n+\frac{2}{5}} + \Lambda_{45}\vartheta_{n+\frac{3}{5}} + \Lambda_{46}\vartheta_{n+\frac{4}{5}} + \Lambda_{47}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+\frac{3}{5}} - \rho_n - \frac{3}{5}h\rho'_n - \frac{\left(\frac{3}{5}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{50}\vartheta_{n-\frac{1}{5}} + \Lambda_{51}\vartheta_{n-\frac{2}{5}} + \Lambda_{52}\vartheta_n + \Lambda_{53}\vartheta_{n+\frac{1}{5}} + \Lambda_{54}\vartheta_{n+\frac{2}{5}} + \Lambda_{55}\vartheta_{n+\frac{3}{5}} + \Lambda_{56}\vartheta_{n+\frac{4}{5}} + \Lambda_{57}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+\frac{4}{5}} - \rho_n - \frac{4}{5}h\rho'_n - \frac{\left(\frac{4}{5}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{60}\vartheta_{n-\frac{1}{5}} + \Lambda_{61}\vartheta_{n-\frac{2}{5}} + \Lambda_{62}\vartheta_n + \Lambda_{63}\vartheta_{n+\frac{1}{5}} + \Lambda_{64}\vartheta_{n+\frac{2}{5}} + \Lambda_{65}\vartheta_{n+\frac{3}{5}} + \Lambda_{66}\vartheta_{n+\frac{4}{5}} + \Lambda_{67}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+1} - \rho_n - h\rho'_n - \frac{(h)^2}{2!}\rho''_n - h^3\left(\Lambda_{70}\vartheta_{n-\frac{1}{5}} + \Lambda_{71}\vartheta_{n-\frac{2}{5}} + \Lambda_{72}\vartheta_n + \Lambda_{73}\vartheta_{n+\frac{1}{5}} + \Lambda_{74}\vartheta_{n+\frac{2}{5}} + \Lambda_{75}\vartheta_{n+\frac{3}{5}} + \Lambda_{76}\vartheta_{n+\frac{4}{5}} + \Lambda_{77}\vartheta_{n+1}\right)
\end{aligned}
\right\}
\tag{11}
$$

$$
\left.
\begin{aligned}
L[\rho'(\xi_n);h] &= \rho'_{n-\frac{1}{5}} - \rho'_n + \frac{1}{5}h\,\rho''_n - h^2 f\left(\mathrm{X}_{110}\vartheta_{n-\frac{1}{5}} + \mathrm{X}_{111}\vartheta_{n-\frac{2}{5}} + \mathrm{X}_{112}\vartheta_n + \mathrm{X}_{113}\vartheta_{n+\frac{1}{5}} + \mathrm{X}_{114}\vartheta_{n+\frac{2}{5}} + \mathrm{X}_{115}\vartheta_{n+\frac{3}{5}} + \mathrm{X}_{116}\vartheta_{n+\frac{4}{5}} + \mathrm{X}_{117}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h] &= \rho'_{n-\frac{2}{5}} - \rho'_n + \frac{2}{5}h\,\rho''_n - h^2\left(\mathrm{X}_{120}\vartheta_{n-\frac{1}{5}} + \mathrm{X}_{121}\vartheta_{n-\frac{2}{5}} + \mathrm{X}_{122}\vartheta_n + \mathrm{X}_{123}\vartheta_{n+\frac{1}{5}} + \mathrm{X}_{124}\vartheta_{n+\frac{2}{5}} + \mathrm{X}_{125}\vartheta_{n+\frac{3}{5}} + \mathrm{X}_{126}\vartheta_{n+\frac{4}{5}} + \mathrm{X}_{127}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h] &= \rho'_{n+\frac{1}{5}} - \rho'_n - \frac{1}{5}h\,\rho''_n - h^2\left(\mathrm{X}_{130}\vartheta_{n-\frac{1}{5}} + \mathrm{X}_{131}\vartheta_{n-\frac{2}{5}} + \mathrm{X}_{132}\vartheta_n + \mathrm{X}_{133}\vartheta_{n+\frac{1}{5}} + \mathrm{X}_{134}\vartheta_{n+\frac{2}{5}} + \mathrm{X}_{135}\vartheta_{n+\frac{3}{5}} + \mathrm{X}_{136}\vartheta_{n+\frac{4}{5}} + \mathrm{X}_{137}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h] &= \rho'_{n+\frac{2}{5}} - \rho'_n - \frac{2}{5}h\,\rho''_n - h^2\left(\mathrm{X}_{140}\vartheta_{n-\frac{1}{5}} + \mathrm{X}_{141}\vartheta_{n-\frac{2}{5}} + \mathrm{X}_{142}\vartheta_n + \mathrm{X}_{143}\vartheta_{n+\frac{1}{5}} + \mathrm{X}_{144}\vartheta_{n+\frac{2}{5}} + \mathrm{X}_{145}\vartheta_{n+\frac{3}{5}} + \mathrm{X}_{146}\vartheta_{n+\frac{4}{5}} + \mathrm{X}_{147}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h] &= \rho'_{n+\frac{3}{5}} - \rho'_n - \frac{3}{5}h\,\rho''_n - h^2\left(\mathrm{X}_{150}\vartheta_{n-\frac{1}{5}} + \mathrm{X}_{151}\vartheta_{n-\frac{2}{5}} + \mathrm{X}_{152}\vartheta_n + \mathrm{X}_{153}\vartheta_{n+\frac{1}{5}} + \mathrm{X}_{154}\vartheta_{n+\frac{2}{5}} + \mathrm{X}_{155}\vartheta_{n+\frac{3}{5}} + \mathrm{X}_{156}\vartheta_{n+\frac{4}{5}} + \mathrm{X}_{157}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h] &= \rho'_{n+\frac{4}{5}} - \rho'_n - \frac{4}{5}h\,\rho''_n - h^2\left(\mathrm{X}_{160}\vartheta_{n-\frac{1}{5}} + \mathrm{X}_{161}\vartheta_{n-\frac{2}{5}} + \mathrm{X}_{162}\vartheta_n + \mathrm{X}_{163}\vartheta_{n+\frac{1}{5}} + \mathrm{X}_{164}\vartheta_{n+\frac{2}{5}} + \mathrm{X}_{165}\vartheta_{n+\frac{3}{5}} + \mathrm{X}_{166}\vartheta_{n+\frac{4}{5}} + \mathrm{X}_{167}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h] &= \rho'_{n+1} - \rho'_n - h\,\rho''_n - h^2\left(\mathrm{X}_{170}\vartheta_{n-\frac{1}{5}} + \mathrm{X}_{171}\vartheta_{n-\frac{2}{5}} + \mathrm{X}_{172}\vartheta_n + \mathrm{X}_{173}\vartheta_{n+\frac{1}{5}} + \mathrm{X}_{174}\vartheta_{n+\frac{2}{5}} + \mathrm{X}_{175}\vartheta_{n+\frac{3}{5}} + \mathrm{X}_{176}\vartheta_{n+\frac{4}{5}} + \mathrm{X}_{177}\vartheta_{n+1}\right)
\end{aligned}
\right\}
\tag{12}
$$

$$
\left.
\begin{aligned}
L[\rho''(\xi_n);h] &= \rho''_{n-\frac{1}{5}} - \rho''_n - h\left(\mathrm{X}_{210}\vartheta_{n-\frac{1}{5}} + \mathrm{X}_{211}\vartheta_{n-\frac{2}{5}} + \mathrm{X}_{212}\vartheta_n + \mathrm{X}_{213}\vartheta_{n+\frac{1}{5}} + \mathrm{X}_{214}\vartheta_{n+\frac{2}{5}} + \mathrm{X}_{215}\vartheta_{n+\frac{3}{5}} + \mathrm{X}_{216}\vartheta_{n+\frac{4}{5}} + \mathrm{X}_{217}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h] &= \rho''_{n-\frac{2}{5}} - \rho''_n - h\left(\mathrm{X}_{220}\vartheta_{n-\frac{1}{5}} + \mathrm{X}_{221}\vartheta_{n-\frac{2}{5}} + \mathrm{X}_{222}\vartheta_n + \mathrm{X}_{223}\vartheta_{n+\frac{1}{5}} + \mathrm{X}_{224}\vartheta_{n+\frac{2}{5}} + \mathrm{X}_{225}\vartheta_{n+\frac{3}{5}} + \mathrm{X}_{226}\vartheta_{n+\frac{4}{5}} + \mathrm{X}_{227}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h] &= \rho''_{n+\frac{1}{5}} - \rho''_n - h\left(\mathrm{X}_{230}\vartheta_{n-\frac{1}{5}} + \mathrm{X}_{231}\vartheta_{n-\frac{2}{5}} + \mathrm{X}_{232}\vartheta_n + \mathrm{X}_{233}\vartheta_{n+\frac{1}{5}} + \mathrm{X}_{234}\vartheta_{n+\frac{2}{5}} + \mathrm{X}_{235}\vartheta_{n+\frac{3}{5}} + \mathrm{X}_{236}\vartheta_{n+\frac{4}{5}} + \mathrm{X}_{237}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h] &= \rho''_{n+\frac{2}{5}} - \rho''_n - h\left(\mathrm{X}_{240}\vartheta_{n-\frac{1}{5}} + \mathrm{X}_{241}\vartheta_{n-\frac{2}{5}} + \mathrm{X}_{242}\vartheta_n + \mathrm{X}_{243}\vartheta_{n+\frac{1}{5}} + \mathrm{X}_{244}\vartheta_{n+\frac{2}{5}} + \mathrm{X}_{245}\vartheta_{n+\frac{3}{5}} + \mathrm{X}_{246}\vartheta_{n+\frac{4}{5}} + \mathrm{X}_{247}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h] &= \rho''_{n+\frac{3}{5}} - \rho''_n - h\left(\mathrm{X}_{250}\vartheta_{n-\frac{1}{5}} + \mathrm{X}_{251}\vartheta_{n-\frac{2}{5}} + \mathrm{X}_{252}\vartheta_n + \mathrm{X}_{253}\vartheta_{n+\frac{1}{5}} + \mathrm{X}_{254}\vartheta_{n+\frac{2}{5}} + \mathrm{X}_{255}\vartheta_{n+\frac{3}{5}} + \mathrm{X}_{256}\vartheta_{n+\frac{4}{5}} + \mathrm{X}_{257}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h] &= \rho''_{n+\frac{4}{5}} - \rho''_n - h\left(\mathrm{X}_{260}\vartheta_{n-\frac{1}{5}} + \mathrm{X}_{261}\vartheta_{n-\frac{2}{5}} + \mathrm{X}_{262}\vartheta_n + \mathrm{X}_{263}\vartheta_{n+\frac{1}{5}} + \mathrm{X}_{264}\vartheta_{n+\frac{2}{5}} + \mathrm{X}_{265}\vartheta_{n+\frac{3}{5}} + \mathrm{X}_{266}\vartheta_{n+\frac{4}{5}} + \mathrm{X}_{267}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h] &= \rho''_{n+1} - \rho''_n - h\left(\mathrm{X}_{270}\vartheta_{n-\frac{1}{5}} + \mathrm{X}_{271}\vartheta_{n-\frac{2}{5}} + \mathrm{X}_{272}\vartheta_n + \mathrm{X}_{273}\vartheta_{n+\frac{1}{5}} + \mathrm{X}_{274}\vartheta_{n+\frac{2}{5}} + \mathrm{X}_{275}\vartheta_{n+\frac{3}{5}} + \mathrm{X}_{276}\vartheta_{n+\frac{4}{5}} + \mathrm{X}_{277}\vartheta_{n+1}\right)
\end{aligned}
\right\}
\tag{13}
$$

3.1.3. Corollary 2

The local truncation error of (8) through (10) is assumed $\rho(\xi)$ to be sufficiently differentiable, and expanding equations (11) through (13) using a Taylor series about ξ_n to obtain

$$L_{-\frac{1}{5}}[\rho(\xi_n);h]=(-5.0968\times10^{-12}),\ L_{-\frac{2}{5}}[\rho(\xi_n);h]=(-2.3806\times10^{-11}),\ L_{\frac{1}{5}}[\rho(\xi_n);h]=(-2.9455\times10^{-12})$$
$$L_{\frac{2}{5}}[\rho(\xi_n);h]=(-1.9538\times10^{-11}),\ L_{\frac{3}{5}}[\rho(\xi_n);h]=(-4.4883\times10^{-11}),\ L_{\frac{4}{5}}[\rho(\xi_n);h]=(-8.4586\times10^{-11})$$
$$,\ L_{1}[\rho(\xi_n);h]=(-1.3308\times10^{-10}),$$

$$L_{-\frac{1}{5}}[\rho'(\xi_n);h]=(9.2176\times10^{-11}),\ L_{-\frac{2}{5}}[\rho'(\xi_n);h]=(-9.3009\times10^{-11}),\ L_{\frac{1}{5}}[\rho'(\xi_n);h]=(-4.7845\times10^{-11})$$
$$L_{\frac{2}{5}}[\rho'(\xi_n);h]=(-1.0384\times10^{-10}),\ L_{\frac{3}{5}}[\rho'(\xi_n);h]=(-1.6800\times10^{-10}),\ L_{\frac{4}{5}}[\rho'(\xi_n);h]=(-1.8782\times10^{-11}),$$
$$L_{1}[\rho'(\xi_n);h]=(-4.8501\times10^{-10}),$$

$$L_{-\frac{1}{5}}[\rho''(\xi_n);h]=(-1.0296\times10^{-09}),\ L_{-\frac{2}{5}}[\rho''(\xi_n);h]=(3.7610\times10^{-09}),\ L_{\frac{1}{5}}[\rho''(\xi_n);h]=(-4.5616\times10^{-10})$$
$$L_{\frac{2}{5}}[\rho''(\xi_n);h]=(-1.0384\times10^{-10}),\ L_{\frac{3}{5}}[\rho''(\xi_n);h]=(-5.6000\times10^{-10}),\ L_{\frac{4}{5}}[\rho''(\xi_n);h]=(-4.6956\times10^{-10}),$$
$$L_{1}[\rho''(\xi_n);h]=(-4.3210\times10^{-09}),$$

3.1.4. Proof

Expanding equations (11) through (13) using Corollary 2 and collecting the like terms to obtain

$$L_{-\frac{1}{5}}[\rho(\xi_n);h]=(-5.0968\times10^{-12})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{11}),\ L_{-\frac{2}{5}}[\rho(\xi_n);h]=(-2.3806\times10^{-11})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{11}),$$
$$L_{\frac{1}{5}}[\rho(\xi_n);h]=(-2.9455\times10^{-12})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{11}),\ L_{\frac{2}{5}}[\rho(\xi_n);h]=(-1.9538\times10^{-11})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{11}),$$
$$L_{\frac{3}{5}}[\rho(\xi_n);h]=(-4.4883\times10^{-11})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{11}),\ L_{\frac{4}{5}}[\rho(\xi_n);h]=(-8.4586\times10^{-11})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{11}),$$
$$L_{1}[\rho(\xi_n);h]=(-1.3308\times10^{-10})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{11}),$$

$$L_{-\frac{1}{5}}[\rho'(\xi_n);h]=(9.2176\times10^{-11})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{10}),\ L_{-\frac{2}{5}}[\rho'(\xi_n);h]=(-9.3009\times10^{-11})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{10}),$$
$$L_{\frac{1}{5}}[\rho'(\xi_n);h]=(-4.7845\times10^{-11})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{10}),\ L_{\frac{2}{5}}[\rho'(\xi_n);h]=(-1.0384\times10^{-10})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{10}),$$
$$L_{\frac{3}{5}}[\rho'(\xi_n);h]=(-1.6800\times10^{-10})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{10}),\ L_{\frac{4}{5}}[\rho'(\xi_n);h]=(-1.8782\times10^{-11})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{10}),$$
$$L_{1}[\rho'(\xi_n);h]=(-4.8501\times10^{-10})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{10}),$$

$$L_{-\frac{1}{5}}[\rho''(\xi_n);h]=(-1.0296\times10^{-09})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{09}),\ L_{-\frac{2}{5}}[\rho''(\xi_n);h]=(3.7610\times10^{-09})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{09}),$$
$$L_{\frac{1}{5}}[\rho''(\xi_n);h]=(-4.5616\times10^{-10})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{09}),\ L_{\frac{2}{5}}[\rho''(\xi_n);h]=(-1.0384\times10^{-10})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{09}),$$
$$L_{\frac{3}{5}}[\rho''(\xi_n);h]=(-5.6000\times10^{-10})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{09}),\ L_{\frac{4}{5}}[\rho''(\xi_n);h]=(-4.6956\times10^{-10})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{09}),$$
$$L_{1}[\rho''(\xi_n);h]=(-4.3210\times10^{-09})C_{08}h^{08}\rho^{08}(\xi_n)+0(h^{09}),$$

3.2. Consistency of NNM

3.2.1. Definition 3

Given the NNM, the first and second characteristic polynomials are defined as,

$$\rho(z)=\sum_{j=0}^{1}\alpha_j z^j \tag{14}$$

$$\sigma(z)=\sum_{j=0}^{1}\beta_j z^j \tag{15}$$

Where z is the principal root, $\alpha_1 \neq 0$ and $(\alpha_0+\beta_0)^2 \neq 0$. The NNM is said to be consistent if it satisfies the following conditions;

i. the order $p \geq 1$,

ii. $\sum_{j=0}^{1}\alpha_j = 0$, and

iii. $\rho(1)=\sigma(1)$

According to definition 3, an LBA is consistent since it is of uniform order eight. Therefore, the LBA satisfies this condition and is deemed consistent.

3.3. Zero Stability of NNM

3.3.1. Definition 4

A NNM Scheme is said to be Zero-stable for any well-behaved problems, provided that

i. All roots of $\rho(r)$ lies in the unit disk, $|r| \leq 1$

ii. Any roots on the unit circle $(|r| \leq 1)$ are simple.

Hence,

$$\rho(u)=\frac{91674240u-1068480u^2+4689496u^3+46746u^4+26397u^5}{58060800-14515200u+604800u^2+151200u^3-15120u^4-630u^5+135u^6} \tag{16}$$

Therefore, equation (16) equal to zero and solving for u gives $u=1$, hence the NNM scheme is zero-stable.

3.4. Convergence of NNM

By Dahlquist's theorem, the necessary and sufficient conditions for NNM to be convergent are that it must be consistent and zero-stable. Therefore, the NNM is convergent, since it is consistent and zero-stable.

3.5. Region of Absolute Stability (RAS) of NNM

To determine the regions of absolute stability of NNM, a method that requires neither the computation of roots of a polynomial nor the solving of simultaneous inequalities was adopted. This method is called the Boundary Locus Method (BLM). The boundary locus method was used to obtain the stability polynomial of NNM as

$$\bar{h}(\pi)=\left(-\frac{3151}{41343750000}\pi^{7}-\frac{10411663}{200037600000000}\pi^{6}\right)h^{7}+\left(-\frac{5358324793}{41343750000}\pi^{6}+\frac{16453}{27562500000}\pi^{7}\right)h^{6}$$
$$+\left(\frac{54159020459}{600112800000000}\pi^{6}+\frac{893657}{74418750000}\pi^{7}\right)h^{5}+\left(\frac{611966338171}{240045120000000}\pi^{6}-\frac{485227}{2205000000}\pi^{7}\right)h^{4}$$
$$+\left(\frac{11944063}{84672000000}\pi^{6}-\frac{1577}{1323000000}\pi^{7}\right)h^{3}+\left(-\frac{13697597}{483840000}\pi^{6}+\frac{529391}{17640000}\pi^{7}\right)h^{2}+\left(-\frac{3}{10}\pi^{6}-\frac{3}{10}\pi^{7}\right)h-\pi^{6}+\pi^{7} \quad (17)$$

Using the stability polynomial in equation (17) to obtain the RAS of NNM as presented in Figure 1.

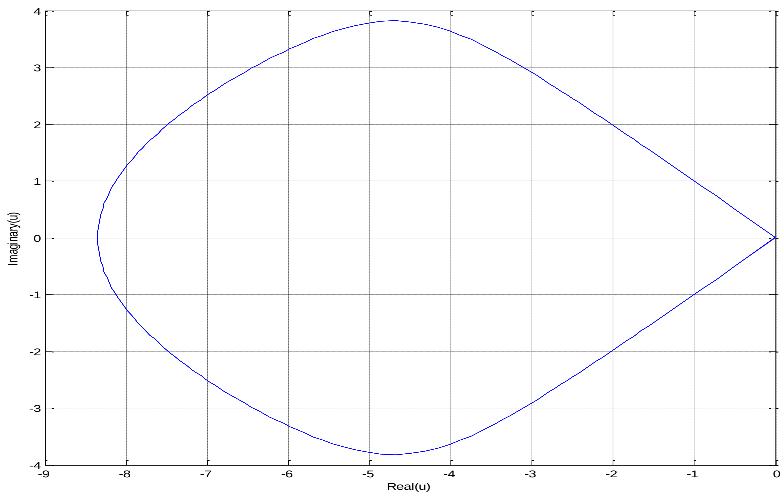


Fig. 1 Regions of absolute stability of NNM

4. Numerical Integration of New Numerical Method (NNM)

The VIDEs of the second were integrated using NNM to test the efficiency and performance of the NNM.

4.1. Example 1

Consider the Volterra integro-differential equation of the form

$$\rho'(\xi)=-\int_0^{\xi}\rho(\tau)d\tau,\ \rho(0)=0,\ 0\le\xi\le1 \quad (18)$$

with the exact solution given by

$$\rho(\xi)=\cos(\xi) \quad (19)$$

Source: [9, 12, 28].

4.2. Example 2

Consider the Volterra integro-differential equation of the form

$$\rho'(\xi)=1+\int_0^{\xi}\rho(\tau)d\tau,\ \rho(0)=0,\ 0\le\xi\le1 \quad (20)$$

with the exact solution given by

$$\rho(\xi) = \sin h(\xi) \tag{21}$$

Source: [11, 12]

4.3. Example 3

Consider the Volterra integro-differential equation of the form

$$\int_0^{\xi} \cos(\xi-\tau)(\rho''(\xi))d\tau = 2\sin(\xi),\ \rho(0)=\rho'(0)=0,\ 0\le\xi\le1 \tag{22}$$

with the exact solution given by

$$\rho(\xi) = \xi^2 \tag{23}$$

Source: [29]

The following notations were used in the tables and figures.

Acronyms	**Meaning**
ξ	- Points of Evaluation
ENNM	- Error in New Numerical Method
EABM5	- Error in Fifth Order Adams-Bashforth-Moulton Predictor-Corrector Method of [28]
E2P3B	- Error in Two Point Three-Step Block Method as in [9]
ETFSM	- Error in Trigonometrically Fitted Simpson's Method of [12]
EGLM	- Error in Third Order General Linear Method of [11]
EHWM	- Error in the Haar Wavelet Method of [29]

Table 1. Numerical results of example 1

ξ	Exact Solution	Computed Solution	ENNM	EABM5	E2P3B
0.025	0.02499739591471233066	0.02499739591471233066	0.0000E00	2.8951E-07	5.7323E-08
0.0125	0.01249967448170978872	0.01249967448170978872	0.0000E00	3.6127E-08	5.5893E-09
0.00625	0.00624995930997530612	0.00624995930997530612	0.0000E00	4.3953E-09	2.2443E-10
0.003125	0.00312499491373946269	0.00312499491373946269	0.0000E00	5.4213E-10	1.3908E-11
0.0015625	0.00156249936421720001	0.00156249936421720001	0.0000E00	6.7325E-11	8.6930E-13
0.00078125	0.00078124992052714272	0.00078124992052714272	0.0000E00	8.3688E-12	5.4179E-14

Table 2. Numerical results of example 2

ξ	Exact Solution	Computed Solution	ENNM	EGLM	ETFSM
0.100	0.10016675001984402582	0.10016675001984402583	1.0000E-20	1.4606E-06	3.7864E-12
0.025	0.02500260424804808603	0.02500260424804808603	0.0000E00	1.6319E-08	2.2030E-16
0.010	0.01000016666750000198	0.01000016666750000198	0.0000E00	8.3870E-10	3.5789E-19
0.005	0.00500002083335937502	0.00500002083335937502	0.0000E00	1.7077E-11	8.0866E-18
0.001	0.00100000016666667500	0.00100000016666667500	0.0000E00	7.2935E-13	-

Table 3. Numerical results of example 3

ξ	Exact Solution	Computed Solution	ENNM	EHWM
8	64.00000000000000000000	64.00000000000000000000	0.0000E00	0.0000E00
16	256. 00000000000000000000	256. 00000000000000000000	0.0000E00	2.2000E-16

32	1024. 00000000000000000000	1024. 00000000000000000000	0.0000E00	4.4000E-16
64	4096. 00000000000000000000	4096. 00000000000000000000	0.0000E00	5.5000E-16
128	16384. 00000000000000000000	16384. 00000000000000000000	0.0000E00	5.5000E-16
256	65536. 00000000000000000000	65536. 00000000000000000000	0.0000E00	8.8000E-16
512	262144. 00000000000000000000	262144. 00000000000000000000	0.0000E00	3.4000E-16
1024	1048576. 00000000000000000000	1048576. 00000000000000000000	0.0000e00	5.5000e-16

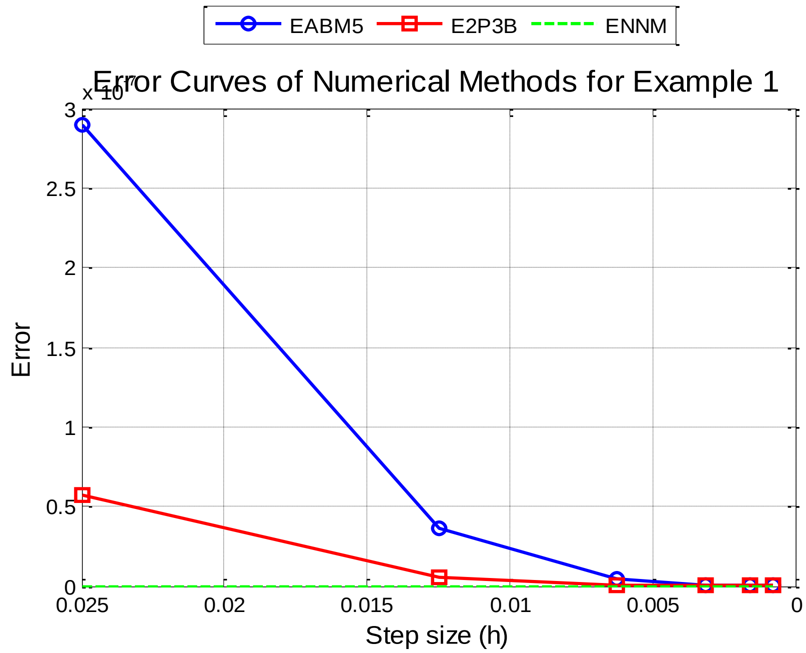


Fig. 2 Textual curve of Table 1

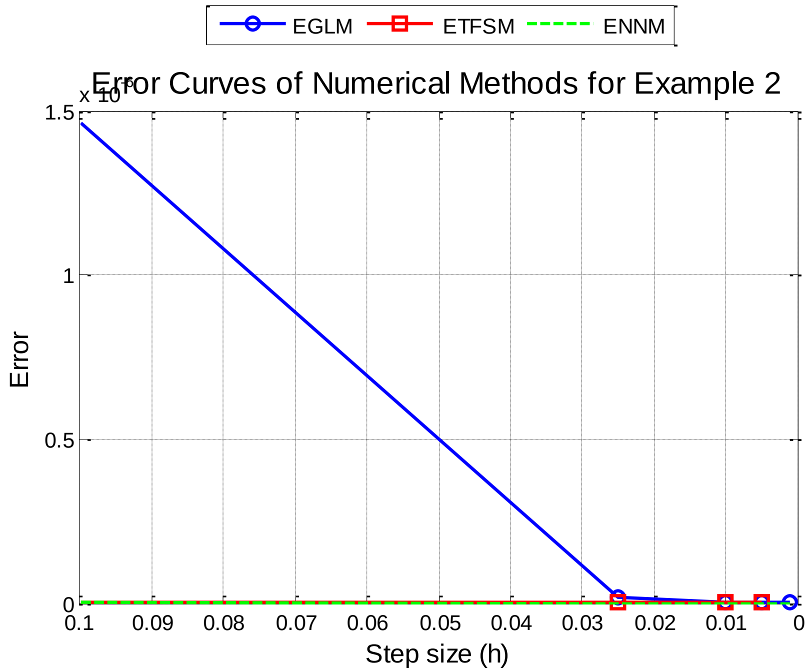


Fig. 3 Textual curve of Table 2

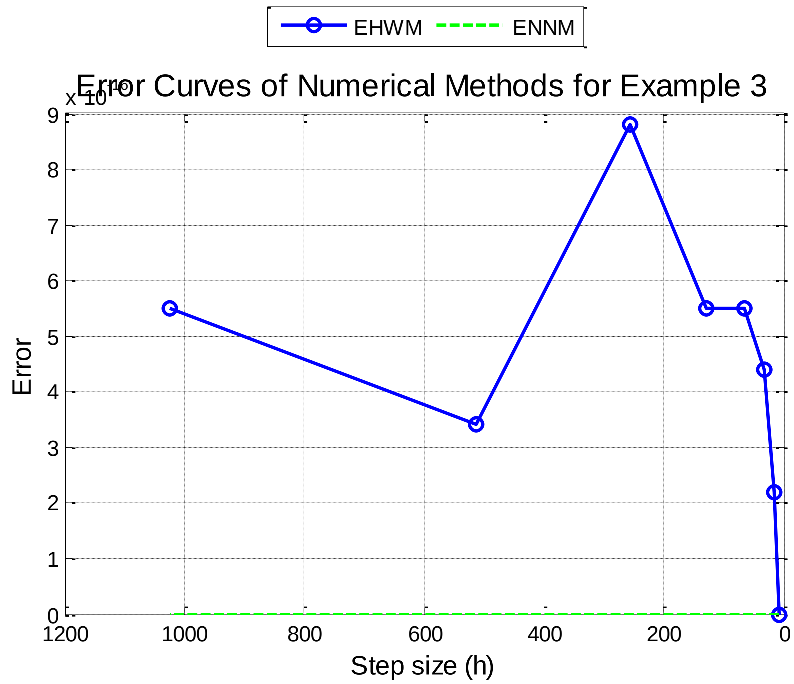


Fig. 4 Textual curve of Table 3

5. Discussion of Results

The New Numerical Method (NNM) was systematically derived using a block algorithm, guided by the general linear multistep framework as presented in Proposition 1. Starting from a single-step method, the block algorithm was applied with higher derivatives to construct an NNM. This approach produced a polynomial formulation that represents the continuous form of the NNM and its higher derivatives, allowing the unknown coefficients to be determined through sequential solution of the system of equations derived from the block algorithm. The derivation ensures that the method captures higher-order behavior of the solution while maintaining a structured and systematic approach to extending single-step methods into a block-based scheme. The resulting generalized NNM provides a framework for high-precision numerical integration of ordinary and integro-differential equations, offering a foundation for further stability and error analysis.

The analytical properties of the NNM were rigorously studied to ensure reliability and accuracy. The order and error constants were determined using linear difference operators and Taylor series expansions, confirming that the method achieves uniform order eight. Consistency was verified by evaluating the characteristic polynomials and confirming the principal root conditions. Zero-stability was established by demonstrating that all roots of the relevant polynomial lie within the unit disk, with simple roots on the unit circle, thereby satisfying the Dahlquist criterion for convergence. Consequently, the NNM is both consistent and zero-stable, guaranteeing convergence for well-behaved problems. Additionally, the region of absolute stability was determined using the Boundary Locus Method, providing a visual representation of the stability boundaries (Figure 1) as a-stable.

The numerical results presented in Tables 1-3 and depicted in Figures 2-4 demonstrate the superior accuracy and robustness of the New Numerical Method (NNM) in solving Volterra integro-differential equations. For Example 1, the NNM achieved exact agreement between the computed and exact solutions at all step sizes, yielding zero error (ENNM). In contrast, the Fifth-Order Adams-Bashforth-Moulton Predictor-Corrector method (EABM5) and the Two-Point Three-Step Block method (E2P3B) exhibited progressively decreasing errors with finer step sizes, as shown in Table 1. Figure 2 illustrates these trends clearly, with the ENNM curve perfectly, while the alternative methods demonstrate minor deviations that diminish as the step size decreases. This underscores the method's stability and high precision for the class of problems represented by Equation (18).

For Example 2, the NNM similarly produced negligible errors across all evaluation points, effectively confirming exactness. Comparatively, the Third-Order General Linear Method (EGLM) and the Trigonometrically Fitted Simpson's Method (ETFSM) exhibited small but measurable deviations, as detailed in Table 2. Figure 3 highlights the distinct separation between the NNM and the other methods. These observations indicate that the NNM maintains superior accuracy even in problems with smaller step sizes and illustrate its robustness in handling integro-differential equations of the form given in Equation (20).

In Example 3**,** the performance of the NNM remained consistently exact, successfully reproducing solutions with exponentially increasing magnitudes across all step sizes, as shown in Table 3. The Haar Wavelet Method (EHWM), while highly accurate, produced minimal non-zero errors. Figure 5 depicts the perfect alignment of the NNM curve with the EHWM, in contrast to the minute deviations of the EHWM. Collectively, these results across all three examples validate the NNM high-order convergence, numerical stability, and superior precision relative to existing methods reported in the literature, confirming its effectiveness for solving Volterra integro-differential equations over a broad range of step sizes and problem scales.

6. Summary Conclusion

This study developed a New Numerical Method (NNM) for the efficient and accurate solution of second-kind Volterra Integro-Differential Equations (VIDEs). The NNM was derived using a block algorithm built upon a generalized linear multistep framework, incorporating higher derivatives and multiple grid points to achieve high-order accuracy. Analytical investigations confirmed the method's consistency, zero-stability, convergence, and a wide region of absolute stability, ensuring its reliability for practical computations. The method was tested on several representative VIDEs, and its performance was compared with that of existing numerical schemes, including the Adams–Bashforth–Moulton predictor-corrector method, general linear methods, trigonometrically fitted schemes, and the Haar wavelet method. Across all examples, the NNM demonstrated superior precision, often producing exact or near-exact solutions with negligible numerical error. These results highlight the NNM's robustness, efficiency, and suitability for solving complex integro-differential equations encountered in scientific, engineering, and applied mathematical contexts.

Authors' Contributions

This research works in collaboration with several authors. "Conceptualization, Y.S.; Methodology, S.J.; Software, S.J.; Validation, D.J.Z.; Formal Analysis, S.J.; Investigation, Y.S.; Resources, D.J.Z.; Writing – Review & Editing, Y.S.. and D.J.Z.

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