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Original Article

Numerical Method for Direct Solution of Bettis and Stiefel Second-Order Oscillatory Differential Equations

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Abstract - This paper introduces a new numerical method for directly solving Bettis and Stiefel second-order oscillatory differential equations, which find extensive use in modeling slowly varying amplitude and phase systems, particularly in physics, engineering, and space orbit mechanics. The majority of conventional numerical methods are found lacking in efficiently solving the above category of equations due to instability and inefficiency in the process of reducing second-order systems to first-order systems. To overcome these restrictions, a recurring hybrid block strategy was constructed with the power series as a base function so that multiple points could be computed at once without system reduction. The technique was tested for its chief numerical properties, including order, consistency, and zero-stability, and it was proved that convergence had a clearly defined region of absolute stability. Numerical examples of Bettis and Stiefel equations demonstrated perfect agreement between the calculated and exact solutions with greater accuracy than current methods. The findings confirm the effectiveness, reliability, and high accuracy of the proposed method for the direct solution of second-order oscillatory systems.

Keywords - Second-order differential equations, Bettis equation, Stiefel equation, Oscillatory systems, Numerical approximation, Power series, Absolute stability.

1. Introduction

For differential equations to deal with periodic or oscillatory systems, it is a major task to accurately accommodate slow changes in amplitude and phase over quite large periods, especially if eigenvalues lie close to the imaginary axis. These indeed represent scenarios where conventional numerical techniques fail in practicality. Hence, researchers have offered a variety of approximations and quasi-envelope strategies so that the method is more accurate. Such considerations assume a prominent focus in real-world applications involving the Bettis and Stiefel Oscillatory Differential Equations, where accurateness of oscillatory behavior is vital [1, 2]. To get past these limitations, one tries to establish more advanced equations: Bettis and Stiefel Oscillatory Differential Equations. The Bettis equation is well-fitted for modeling systems with slowly changing oscillatory behavior in physics and engineering, whilst the Stiefel equation is fit for high-dimensional systems and provides an excellent framework for matrix differential equations and eigenvalue tracking in orbital mechanics and structural analysis [3]. The study aims to set up numerical procedures for dealing with oscillatory solutions for second-order differential equations in the form:

$$\xi''(\tau) = f(\tau, \xi, \xi'), \xi(\tau_0) = \eta_0, \xi'(\tau_1) = \eta_1' \tag{1}$$

Where τ_0 represents the initial value/point, ξ_0 denotes the solution at time τ_0, and f remains continuous over the integration interval.

Classical procedures were employed to solve second-order differential equations by reducing them to a first-order system, wherein the cost and possibility of numerical inaccuracies escalate. To overcome these disadvantages, block hybrid methods have been developed that integrate second-order initial value problems directly and provide better efficiency and accuracy [4, 5]. Thus, we are employing such methods that are useful in modeling oscillatory behavior in engineering and economics [6, 7]. Whereas ref. [6, 8, 9] took this one step further by developing these methods to estimate the solution at several points simultaneously, new innovations such as the double-step hybrid methods of [10-12] present yet another demonstration of how flexible and capable block hybrid methods are for dealing with complicated ODEs arising in scientific applications.

That paper [1, 2] presents numerical solutions to second-order stiff and oscillatory differential equations, specifically the Bettis and Stiefel system-related equations. [1] presents a hybrid multistep method that is somehow more efficient than earlier schemes; it further addresses a second-order differential equation directly instead of converting it, and works exceptionally well in stiff and oscillatory situations. Conversely, [2] suggested the development of a hybrid block approach of remarkable accuracy, showing an absolute error of zero for all Bettis and Stiefel test problems. The two above approaches present a hybrid numerical methodology as the tool to handle complex dynamical systems of oscillatory character, especially those represented by Bettis and Stiefel differential equations.

2. Materials and Methods

To derive this new method, we use a power series polynomial and a constructed approximated solution is expressed in the form:

$$\xi(\tau)=\sum_{j=0}^{u+v-1} a_j \tau^j \tag{2}$$

Equation (2) is used as the basis function to estimate the solution of (1), with its second derivative of the form

$$\xi''(\tau)=\sum_{j=0}^{u+v-1} j(j-1)a_j \tau^{j-2} \tag{3}$$

Where $\tau \in [0,1]$ the $a's$ real unknown parameters are to be determined, and $u+v$ is the sum of the number of interpolation and collocation points. Let the solution of (1) be sought on the partition $\pi_N : 0=\tau_0<\tau_1<\tau_2<\Lambda\ \tau_n<\tau_{n+1}<\tau_N=1$ of the interval $[0,1]$ with a constant step size h, given by $h=\tau_n-\tau_{n-1}$, where $n=0,1,2,...,N$.

Equation (3) is substituted into (1) to give

$$\sum_{j=0}^{u+v-1} j(j-1)a_j \tau^{j-2} = f(\tau,\xi,\xi') \tag{4}$$

Evaluate (2) at $u=2$ and $v=7$ to gives the polynomial of degree $u+v-1$ as follows

$$\xi(\tau)=\sum_{j=0}^{8} a_j \tau^j \tag{5}$$

Differentiate equation (5) twice to get

$$\sum_{j=0}^{8} j(j-1)a_j \tau^{j-2} = f(\tau, \xi, \xi') \tag{6}$$

Therefore, Equation (5) is interpolated at the point $\tau_{n+u}, u=\frac{1}{2},1$ and Equation (6) is collocated $\tau_{n+v}, v=0,\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4},1$ to give a system of nonlinear equations in a matrix form as

$$\begin{bmatrix}
1 & \tau_{n+\frac{1}{2}} & \tau^2_{n+\frac{1}{2}} & \tau^3_{n+\frac{1}{2}} & \tau^4_{n+\frac{1}{2}} & \tau^5_{n+\frac{1}{2}} & \tau^6_{n+\frac{1}{2}} & \tau^7_{n+\frac{1}{2}} & \tau^8_{n+\frac{1}{2}} \\
1 & \tau_{n+1} & \tau^2_{n+1} & \tau^3_{n+1} & \tau^4_{n+1} & \tau^5_{n+1} & \tau^6_{n+1} & \tau^7_{n+1} & \tau^8_{n+1} \\
0 & 0 & 2 & 6\tau_n & 12\tau_n^2 & 20\tau_n^3 & 30\tau_n^4 & 42t_n^5 & 56\tau_n^6 \\
0 & 0 & 2 & 6\tau_{n+\frac{1}{4}} & 12\tau^2_{n+\frac{1}{4}} & 20\tau^3_{n+\frac{1}{4}} & 30\tau^4_{n+\frac{1}{4}} & 42\tau^5_{n+\frac{1}{4}} & 56\tau^6_{n+\frac{1}{4}} \\
0 & 0 & 2 & 6\tau_{n+\frac{1}{3}} & 12\tau^2_{n+\frac{1}{3}} & 20\tau^3_{n+\frac{1}{3}} & 30\tau^4_{n+\frac{1}{3}} & 42\tau^5_{n+\frac{1}{3}} & 56\tau^6_{n+\frac{1}{3}} \\
0 & 0 & 2 & 6\tau_{n+\frac{1}{2}} & 12\tau^2_{n+\frac{1}{2}} & 20\tau^3_{n+\frac{1}{2}} & 30\tau^4_{n+\frac{1}{2}} & 42\tau^5_{n+\frac{1}{2}} & 56\tau^6_{n+\frac{1}{2}} \\
0 & 0 & 2 & 6\tau_{n+\frac{2}{3}} & 12\tau^2_{n+\frac{2}{3}} & 20\tau^3_{n+\frac{2}{3}} & 30\tau^4_{n+\frac{2}{3}} & 42\tau^5_{n+\frac{2}{3}} & 56\tau^6_{n+\frac{2}{3}} \\
0 & 0 & 2 & 6\tau_{n+\frac{3}{4}} & 12\tau^2_{n+\frac{3}{4}} & 20\tau^3_{n+\frac{3}{4}} & 30\tau^4_{n+\frac{3}{4}} & 42\tau^5_{n+\frac{3}{4}} & 56\tau^6_{n+\frac{3}{4}} \\
0 & 0 & 2 & 6\tau_{n+1} & 12t^2_{n+1} & 20\tau^3_{n+1} & 30\tau^4_{n+1} & 42\tau^5_{n+1} & 56\tau^6_{n+1}
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix}
=
\begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ f_n \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{2}{3}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{bmatrix} \tag{7}$$

The Gaussian elimination method was employed on Equation (7) to get the unknown values $a_j{}'s$. After expanding (7), the values of $a_j{}'s$ are then substituted into (8), after some manipulations to get the continuous form of the scheme as

$$\xi(\tau)=\alpha_{\frac{1}{2}}(\tau)\xi_{n+\frac{1}{2}}+\alpha_1(\tau)\xi_{n+1}+h^2\left[\sum_{j=0}^{1}\beta_j(\tau)f_{n+j}+\beta_{v_i}(\tau)f_{n+v_i}\right], v_i=0,\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4} \tag{8}$$

The values of $\alpha_{\frac{1}{2}},\alpha_1,\beta_0,\beta_{\frac{1}{4}},\beta_{\frac{1}{3}},\beta_{\frac{1}{2}},\beta_{\frac{2}{3}},\beta_{\frac{3}{4}},\beta_1$ are,

$$\alpha_{\frac{1}{2}} = 2 - 2\tau$$
$$\alpha_1 = 2\tau - 1$$
$$\beta_0 = \frac{41}{13440} - \frac{2539}{40320}\tau + \frac{1}{2}\tau^2 - \frac{77}{36}\tau^3 + \frac{49}{9}\tau^4 - \frac{203}{24}\tau^5 + \frac{707}{90}\tau^6 - 4t^7 + \frac{6}{7}\tau^8$$
$$\beta_{\frac{1}{4}} = \frac{2}{21} - \frac{74}{103}\tau + \frac{256}{15}\tau^3 - \frac{3392}{45}\tau^4 + \frac{768}{5}\tau^5 - \frac{7552}{45}\tau^6 + \frac{3328}{35}\tau^7 - \frac{768}{35}\tau^8$$
$$\beta_{\frac{1}{3}} = -\frac{243}{4480} + \frac{81}{128}\tau - \frac{243}{10}\tau^3 + \frac{4779}{40}\tau^4 - \frac{10449}{40}\tau^5 + \frac{2997}{10}\tau^6 - \frac{6156}{35}\tau^7 + \frac{1458}{35}\tau^8$$
$$\beta_{\frac{1}{2}} = \frac{17}{105} - \frac{23}{35}\tau + 16\tau^3 - \frac{260}{3}\tau^4 + \frac{1048}{5}\tau^5 - \frac{3928}{15}\tau^6 + \frac{1152}{7}\tau^7 - \frac{288}{7}\tau^8$$
$$\beta_{\frac{2}{3}} = -\frac{243}{4480} + \frac{1539}{4480}\tau - \frac{243}{20}\tau^3 + \frac{1377}{20}\tau^4 - \frac{7047}{40}\tau^5 + \frac{2349}{10}\tau^6 - \frac{5508}{35}\tau^7 + \frac{1458}{35}\tau^8$$
$$\beta_{\frac{3}{4}} = \frac{2}{21} - \frac{94}{315}\tau + \frac{256}{45}\tau^3 - \frac{1472}{45}\tau^4 + \frac{256}{3}\tau^5 - \frac{5248}{45}\tau^6 + \frac{2816}{35}\tau^7 - \frac{768}{35}\tau^8$$
$$\beta_1 = \frac{41}{13440} - \frac{41}{13440}\tau - \frac{1}{6}\tau^3 + \frac{71}{72}\tau^4 - \frac{107}{40}\tau^5 + \frac{347}{90}\tau^6 - \frac{20}{7}\tau^7 + \frac{6}{7}\tau^8$$

Simplify (8) at non-interpolating points to get,

$$\left.\begin{aligned}
&\xi_n - 2y_{n+\frac{1}{2}} + \xi_{n+1} = \frac{41}{13440}h^2 f_n + \frac{2}{21}h^2 f_{n+\frac{1}{4}} - \frac{243}{4480}h^2 f_{n+\frac{1}{3}} + \frac{17}{105}h^2 f_{n+\frac{1}{2}} - \frac{243}{4480}h^2 f_{n+\frac{2}{3}} + \frac{2}{21}h^2 f_{n+\frac{3}{4}} + \frac{41}{13440} f_{n+1}\\
&\xi_{n+\frac{1}{4}} - \frac{3}{2}\xi_{n+\frac{1}{2}} + \frac{1}{2}\xi_{n+1} = -\frac{193}{1146880}h^2 f_n + \frac{31}{5376}h^2 f_{n+\frac{1}{4}} - \frac{243}{229376}h^2 f_{n+\frac{1}{3}} + \frac{4247}{71680}h^2 f_{n+\frac{1}{2}} - \frac{14823}{1146880}h^2 f_{n+\frac{2}{3}} - \frac{369}{8960}h^2 f_{n+\frac{3}{4}} + \frac{5861}{3440640}h^2 f_{n+1}\\
&\xi_{n+\frac{1}{3}} - \frac{4}{3}\xi_{n+\frac{1}{2}} + \frac{1}{3}\xi_{n+1} = -\frac{1727}{14696640}h^2 f_n + \frac{884}{229635}h^2 f_{n+\frac{1}{4}} - \frac{379}{60480}h^2 f_{n+\frac{1}{3}} + \frac{2894}{76545}h^2 f_{n+\frac{1}{2}} - \frac{491}{60480}h^2 f_{n+\frac{2}{3}} + \frac{1252}{45927}h^2 f_{n+\frac{3}{4}} + \frac{16753}{14696640}h^2 f_{n+1}\\
&\xi_n - \frac{2}{3}\xi_{n+\frac{1}{2}} - \frac{1}{3}\xi_{n+1} = \frac{3617}{29393280}h^2 f_n - \frac{206}{45927}h^2 f_{n+\frac{1}{4}} + \frac{241}{24192}h^2 f_{n+\frac{1}{3}} - \frac{1237}{76545}h^2 f_{n+\frac{1}{2}} + \frac{1429}{120960}h^2 f_{n+\frac{2}{3}} - \frac{6406}{229635}h^2 f_{n+\frac{3}{4}} + \frac{33343}{293393280} f_{n+1}\\
&\xi_n - \frac{1}{2}\xi_{n+\frac{1}{2}} + \frac{1}{2}\xi_{n+1} = -\frac{613}{3440640}h^2 f_n - \frac{173}{26880}h^2 f_{n+\frac{1}{4}} + \frac{16281}{1146880}h^2 f_{n+\frac{1}{3}} + \frac{4667}{215040}h^2 f_{n+\frac{1}{2}} + \frac{29889}{1146880}h^2 f_{n+\frac{2}{3}} - \frac{75}{1792}h^2 f_{n+\frac{3}{4}} - \frac{5827}{3440640} f_{n+1}
\end{aligned}\right\} \quad (9)$$

Equation (8) is differentiated once as,

$$\xi'(\tau) = \sigma_{\frac{1}{2}}(\tau)\xi_{n+\frac{1}{2}} + \sigma_1(\tau)\xi_{n+1} + h^2\left[\sum_{j=0}^{1}\delta_j(\tau) f_{n+j} + \delta_v(\tau) f_{n+v_i}\right] \quad (10)$$

The unknown values of (10) are,

$$\sigma_{\frac{1}{2}} = -2$$
$$\sigma_1 = 2$$
$$\delta_0 = -\frac{2539}{40320} + \tau - \frac{77}{12}\tau^2 + \frac{196}{9}\tau^3 - \frac{1015}{24}\tau^4 + \frac{707}{15}\tau^5 - 28\tau^6 + \frac{48}{7}\tau^7$$
$$\delta_{\frac{1}{4}} = -\frac{74}{105} + \frac{256}{5}\tau^2 - \frac{13568}{45}\tau^3 + 768\tau^4 - \frac{15105}{15}\tau^5 + \frac{3328}{5}\tau^6 - \frac{6144}{35}\tau^7$$
$$\delta_{\frac{1}{3}} = \frac{81}{128} - \frac{729}{10}\tau^2 + \frac{4779}{10}\tau^3 - \frac{10449}{8}\tau^4 + \frac{8991}{5}\tau^5 - \frac{6156}{5}\tau^6 + \frac{11664}{35}\tau^7$$
$$\delta_{\frac{1}{2}} = -\frac{23}{35} + 48\tau^2 - \frac{1040}{3}\tau^3 + 1048\tau^4 - \frac{7856}{5}\tau^5 + 1152\tau^6 - \frac{2304}{7}\tau^7$$
$$\delta_{\frac{2}{3}} = \frac{1539}{4480} - \frac{729}{20}\tau^2 + \frac{1377}{5}\tau^3 - \frac{7047}{8}\tau^4 + \frac{7047}{5}\tau^5 - \frac{5508}{5}\tau^6 + \frac{11664}{35}\tau^7$$
$$\delta_{\frac{3}{4}} = -\frac{94}{315} + \frac{256}{15}\tau^2 - \frac{5888}{45}\tau^3 + \frac{1280}{3}\tau^4 - \frac{10496}{15}\tau^5 + \frac{2816}{5}\tau^6 - \frac{6144}{35}\tau^7$$
$$\delta_1 = -\frac{41}{13440} - \frac{1}{2}\tau^2 + \frac{71}{18}\tau^3 - \frac{107}{8}\tau^4 + \frac{347}{15}\tau^5 - 20\tau^6 + \frac{48}{7}\tau^7$$

Evaluating (10) at all points, to get the discrete schemes is as,

$$\left.\begin{aligned}
&h\xi'_{n}+2\xi_{n+\frac{1}{2}}-2\xi_{n+1}=-\frac{2539}{40320}hf_{n}-\frac{74}{105}hf_{n+\frac{1}{4}}+\frac{81}{128}hf_{n+\frac{1}{3}}+\frac{23}{25}hf_{n+\frac{1}{2}}+\frac{1539}{4480}hf_{n+\frac{2}{3}}-\frac{94}{315}hf_{n+\frac{3}{4}}+\frac{41}{13440}hf_{n+1}\\
&h\xi'_{n+\frac{1}{4}}+2\xi_{n+\frac{1}{2}}-2\xi_{n+1}=\frac{29}{43008}hf_{n}-\frac{239}{5040}hf_{n+\frac{1}{4}}-\frac{5913}{71680}hf_{n+\frac{1}{3}}-\frac{851}{3360}hf_{n+\frac{1}{2}}+\frac{567}{10240}hf_{n+\frac{2}{3}}-\frac{93}{560}hf_{n+\frac{3}{4}}-\frac{4379}{645120}hf_{n+1}\\
&h\xi'_{n+\frac{1}{3}}+2\xi_{n+\frac{1}{2}}-2\xi_{n+1}=-\frac{259}{466560}hf_{n}-\frac{106}{8505}hf_{n+\frac{1}{4}}-\frac{1157}{40320}hf_{n+\frac{1}{3}}+\frac{2221}{8505}hf_{n+\frac{1}{2}}+\frac{2411}{40320}hf_{n+\frac{2}{3}}-\frac{4286}{25515}hf_{n+\frac{3}{4}}+\frac{7337}{1088640}hf_{n+1}\\
&h\xi'_{n+\frac{1}{2}}+2\xi_{n+\frac{1}{2}}-2\xi_{n+1}=\frac{17}{20160}hf_{n}-\frac{2}{63}hf_{n+\frac{1}{4}}+\frac{81}{1120}hf_{n+\frac{1}{3}}-\frac{17}{105}hf_{n+\frac{1}{2}}+\frac{81}{2240}hf_{n+\frac{2}{3}}-\frac{10}{63}hf_{n+\frac{3}{4}}-\frac{1}{144}hf_{n+1}\\
&h\xi'_{n+\frac{2}{3}}+2\xi_{n+\frac{1}{2}}-2\xi_{n+1}=-\frac{139}{217728}hf_{n}+\frac{82}{3645}hf_{n+\frac{1}{4}}+\frac{1963}{40320}hf_{n+\frac{1}{3}}-\frac{533}{8505}hf_{n+\frac{1}{2}}+\frac{5531}{40320}hf_{n+\frac{2}{3}}-\frac{1514}{8505}hf_{n+\frac{3}{4}}-\frac{21739}{3265920}hf_{n+1}\\
&h\xi'_{n+\frac{3}{4}}+2\xi_{n+\frac{1}{2}}-2\xi_{n+1}=-\frac{443}{645120}hf_{n}\ \frac{41}{1680}hf_{n+\frac{1}{4}}+\frac{3807}{71680}hf_{n+\frac{1}{3}}-\frac{79}{1120}hf_{n+\frac{1}{2}}+\frac{13689}{71680}hf_{n+\frac{2}{3}}-\frac{103}{720}hf_{n+\frac{3}{4}}-\frac{1457}{215040}hf_{n+1}\\
&h\xi'_{n+1}+2\xi_{n+\frac{1}{2}}-2\xi_{n+1}=-\frac{41}{13440}hf_{n}+\frac{34}{315}hf_{n+\frac{1}{4}}-\frac{1053}{4480}hf_{n+\frac{1}{3}}+\frac{1}{3}hf_{n+\frac{1}{2}}-\frac{2349}{4480}hf_{n+\frac{2}{3}}+\frac{18}{35}hf_{n+\frac{3}{4}}+\frac{2293}{40320}hf_{n+1}
\end{aligned}\right\} \tag{11}$$

Equations (9) and (11) are combined to yield the computational hybrid block method, which can be written explicitly as,

$$\left.\begin{aligned}
&\xi_{n+\frac{1}{4}}=\xi_{n}+\frac{1}{4}h\xi'_{n}+\frac{129271}{10321920}h^{2}f_{n}+\frac{111}{1280}h^{2}f_{n+\frac{1}{4}}-\frac{120447}{1146880}h^{2}f_{n+\frac{1}{3}}+\frac{13253}{215040}h^{2}f_{n+\frac{1}{2}}-\frac{51111}{1146880}h^{2}f_{n+\frac{2}{3}}+\frac{1657}{80640}h^{2}f_{n+\frac{3}{4}}-\frac{2011}{3440640}h^{2}f_{n+1}\\
&\xi_{n+\frac{1}{3}}=\xi_{n}+\frac{1}{3}h\xi'_{n}+\frac{32741}{1837080}h^{2}f_{n}+\frac{6592}{45927}h^{2}f_{n+\frac{1}{4}}-\frac{22}{135}h^{2}f_{n+\frac{1}{3}}+\frac{7268}{76545}h^{2}f_{n+\frac{1}{2}}-\frac{517}{7560}h^{2}f_{n+\frac{2}{3}}+\frac{7232}{229635}h^{2}f_{n+\frac{3}{4}}-\frac{821}{918540}h^{2}f_{n+1}\\
&\xi_{n+\frac{1}{2}}=\xi_{n}+\frac{1}{2}h\xi'_{n}+\frac{2293}{80640}h^{2}f_{n}+\frac{9}{35}h^{2}f_{n+\frac{1}{4}}-\frac{2349}{8960}h^{2}f_{n+\frac{1}{3}}+\frac{1}{6}h^{2}f_{n+\frac{1}{2}}-\frac{1053}{8960}h^{2}f_{n+\frac{2}{3}}+\frac{17}{315}h^{2}f_{n+\frac{3}{4}}-\frac{41}{26880}h^{2}f_{n+1}\\
&\xi_{n+\frac{2}{3}}=\xi_{n}+\frac{2}{3}h\xi'_{n}+\frac{8968}{229635}h^{2}f_{n}+\frac{84992}{229635}h^{2}f_{n+\frac{1}{4}}-\frac{338}{945}h^{2}f_{n+\frac{1}{3}}+\frac{19904}{76545}h^{2}f_{n+\frac{1}{2}}-\frac{22}{135}h^{2}f_{n+\frac{2}{3}}+\frac{17408}{229635}h^{2}f_{n+\frac{3}{4}}-\frac{494}{229635}h^{2}f_{n+1}\\
&\xi_{n+\frac{3}{4}}=\xi_{n}+\frac{3}{4}h\xi'_{n}+\frac{50871}{1146880}h^{2}f_{n}+\frac{765}{1792}h^{2}f_{n+\frac{1}{4}}-\frac{465831}{1146880}h^{2}f_{n+\frac{1}{3}}+\frac{22167}{71680}h^{2}f_{n+\frac{1}{2}}-\frac{203391}{1146880}h^{2}f_{n+\frac{2}{3}}+\frac{111}{1280}h^{2}f_{n+\frac{3}{4}}-\frac{2817}{1146880}h^{2}f_{n+1}\\
&\xi_{n+1}=\xi_{n}+h\xi'_{n}+\frac{151}{2520}h^{2}f_{n}+\frac{64}{105}h^{2}f_{n+\frac{1}{4}}-\frac{81}{140}h^{2}f_{n+\frac{1}{3}}+\frac{52}{105}h^{2}f_{n+\frac{1}{2}}-\frac{81}{280}h^{2}f_{n+\frac{2}{3}}+\frac{64}{315}h^{2}f_{n+\frac{3}{4}}\\
&\xi'_{n+\frac{1}{4}}=\xi'_{n}+\frac{41059}{645120}hf_{n}+\frac{3313}{5040}hf_{n+\frac{1}{4}}-\frac{51273}{71680}hf_{n+\frac{1}{3}}+\frac{1357}{3360}hf_{n+\frac{1}{2}}-\frac{4131}{14336}hf_{n+\frac{2}{3}}+\frac{667}{5040}hf_{n+\frac{3}{4}}-\frac{2411}{645120}hf_{n+1}\\
&\xi'_{n+\frac{1}{3}}=y'_{n}+\frac{12967}{204120}hf_{n}+\frac{5888}{8505}hf_{n+\frac{1}{4}}-\frac{1667}{2520}hf_{n+\frac{1}{3}}+\frac{3368}{8505}hf_{n+\frac{1}{2}}-\frac{143}{504}hf_{n+\frac{2}{3}}+\frac{3328}{25515}hf_{n+\frac{3}{4}}-\frac{251}{68040}hf_{n+1}\\
&\xi'_{n+\frac{1}{2}}=\xi'_{n}+\frac{2573}{40320}hf_{n}+\frac{212}{315}hf_{n+\frac{1}{4}}-\frac{2511}{4480}hf_{n+\frac{1}{3}}+\frac{52}{105}hf_{n+\frac{1}{2}}-\frac{1377}{4480}hf_{n+\frac{2}{3}}+\frac{44}{315}hf_{n+\frac{3}{4}}-\frac{157}{40320}hf_{n+1}\\
&\xi'_{n+\frac{2}{3}}=\xi'_{n}+\frac{541}{8505}hf_{n}+\frac{17408}{25515}hf_{n+\frac{1}{4}}-\frac{184}{315}hf_{n+\frac{1}{3}}+\frac{5056}{8505}hf_{n+\frac{1}{2}}-\frac{13}{63}hf_{n+\frac{2}{3}}+\frac{1024}{8505}hf_{n+\frac{3}{4}}-\frac{92}{25515}hf_{n+1}\\
&\xi'_{n+\frac{3}{4}}=\xi'_{n}+\frac{4563}{71680}hf_{n}+\frac{381}{560}hf_{n+\frac{1}{4}}-\frac{41553}{71680}hf_{n+\frac{1}{3}}+\frac{657}{1120}hf_{n+\frac{1}{2}}-\frac{2187}{14336}hf_{n+\frac{2}{3}}+\frac{87}{560}hf_{n+\frac{3}{4}}-\frac{267}{71680}hf_{n+1}\\
&\xi'_{n+1}=\xi'_{n}+\frac{151}{2520}hf_{n}+\frac{256}{315}hf_{n+\frac{1}{4}}-\frac{243}{280}hf_{n+\frac{1}{3}}+\frac{104}{105}hf_{n+\frac{1}{2}}-\frac{280}{243}hf_{n+\frac{2}{3}}+\frac{256}{315}hf_{n+\frac{3}{4}}-\frac{151}{2520}hf_{n+1}
\end{aligned}\right\} \tag{12}$$

3. Basic Properties of the Method

The conditions of basic properties of the new method are numerically analysed.

3.1. Order and Error Constant

The linear operator associated in L linked with Equation (12) is defined as

$$L[\xi(\tau);h]=\sum_{j=0}^{1}\left\{\alpha_{u}\xi(\tau_{n}+uh)-h^{2}\beta_{v}\xi^{2}(\tau_{n}+vh)\right\} \tag{13}$$

Where $\xi(\tau)$ an arbitrary test function is continuously differentiable in the interval $[a, b]$. Expand Equation (13) using a Taylor series by τ_n collecting like terms h and y to obtain the expression.

$$\lambda\{\xi(\tau);h\} = C_0\xi(\tau) + C_1\xi'(\tau) + \ldots + C_p h^p \xi^p(\tau) + C_{p+1}h^{p+1}\xi^{p+1}(\tau) + C_{p+2}h^{p+2}\xi^{p+2}(\tau) + \ldots \tag{14}$$

Using the linear operator $L[\xi(\tau_n);h]$ in (13) with the Corollaries 1 and 2 below to get the order and error constant of (12) [2].

3.1.1. Corollary 1

The operator defined in (13) is linked with the local truncation error of (12) $C_{07}h^{06}\xi^{06}(\tau_n) + 0(h^{08})$.

Proof

The linear difference operators associated with (12) are

$$\left.\begin{array}{l}
L[\xi(\tau_n);h] = y\left(\tau_n + \frac{1}{4}h\right) - \left[\sum_{i=u}(\alpha_u(\tau_n + uh)) + \sum_{i=v}(\beta_v(\tau) f_{n+v})\right] \\
L[\xi(\tau_n);h] = y\left(\tau_n + \frac{1}{3}h\right) - \left[\sum_{i=u}(\alpha_u(\tau_n + uh)) + \sum_{i=v}(\beta_v(\tau) f_{n+v})\right] \\
L[\xi(\tau_n);h] = y\left(\tau_n + \frac{1}{2}h\right) - \left[\sum_{i=u}(\alpha_u(\tau_n + uh)) + \sum_{i=v}(\beta_v(\tau) f_{n+v})\right] \\
L[\xi(\tau_n);h] = y\left(\tau_n + \frac{2}{3}h\right) - \left[\sum_{i=u}(\alpha_u(\tau_n + uh)) + \sum_{i=v}(\beta_v(\tau) f_{n+v})\right] \\
L[\xi(\tau_n);h] = y\left(\tau_n + \frac{3}{4}h\right) - \left[\sum_{i=u}(\alpha_u(\tau_n + uh)) + \sum_{i=v}(\beta_v(\tau) f_{n+v})\right] \\
L[\xi(\tau_n);h] = y(\tau_n + h) - \left[\sum_{i=u}(\alpha_u(\tau_n + uh)) + \sum_{i=v}(\beta_v(\tau) f_{n+v})\right]
\end{array}\right\} \tag{15}$$

Substituting (12) in Equation (14) into (15) to get

$$\left[\begin{array}{l}
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{4}\right)^j}{j!} - \xi_n - \frac{1}{4}h\xi'_n - \frac{129271}{10321920}h\xi''_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}\xi_n^{j+2}\left[\frac{111}{1280}\left(\frac{1}{4}\right) - \frac{120447}{1146880}\left(\frac{1}{3}\right) + \frac{13253}{215040}\left(\frac{1}{2}\right) - \frac{51111}{1146880}\left(\frac{2}{3}\right) + \frac{1657}{80640}\left(\frac{3}{4}\right) - \frac{2011}{3440640}(1)\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{3}\right)^j}{j!} - \xi_n - \frac{1}{3}h\xi'_n - \frac{32741}{1837080}h\xi''_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}\xi_n^{j+2}\left[\frac{6592}{45927}\left(\frac{1}{4}\right) - \frac{22}{135}\left(\frac{1}{3}\right) + \frac{7268}{76545}\left(\frac{1}{2}\right) - \frac{517}{7560}\left(\frac{2}{3}\right) + \frac{7232}{229635}\left(\frac{3}{4}\right) - \frac{821}{918540}(1)\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)^j}{j!} - \xi_n - \frac{1}{2}h\xi'_n - \frac{2293}{80640}h\xi''_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}\xi_n^{j+2}\left[\frac{9}{35}\left(\frac{1}{4}\right) - \frac{2349}{8960}\left(\frac{1}{3}\right) + \frac{1}{6}\left(\frac{1}{2}\right) - \frac{1053}{8960}\left(\frac{2}{3}\right) + \frac{17}{315}\left(\frac{3}{4}\right) - \frac{41}{26880}(1)\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{2}{3}\right)^j}{j!} - \xi_n - \frac{2}{3}h\xi'_n - \frac{8968}{229635}h\xi''_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}\xi_n^{j+2}\left[\frac{84992}{229635}\left(\frac{1}{4}\right) - \frac{338}{945}\left(\frac{1}{3}\right) + \frac{19904}{76545}\left(\frac{1}{2}\right) - \frac{22}{135}\left(\frac{2}{3}\right) + \frac{17408}{229635}\left(\frac{3}{4}\right) - \frac{494}{229635}(1)\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{3}{4}\right)^j}{j!} - \xi_n - \frac{3}{4}h\xi'_n - \frac{50871}{1146880}h\xi''_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}\xi_n^{j+2}\left[\frac{765}{1792}\left(\frac{1}{4}\right) - \frac{465831}{1146880}\left(\frac{1}{3}\right) + \frac{22167}{71680}\left(\frac{1}{2}\right) - \frac{203391}{1146880}\left(\frac{2}{3}\right) + \frac{111}{1280}\left(\frac{3}{4}\right) - \frac{2817}{1146880}(1)\right] \\
\sum_{j=0}^{\infty}\frac{(1)^j}{j!} - \xi_n - h\xi'_n - \frac{151}{2520}h\xi''_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}\xi_n^{j+2}\left[\frac{64}{105}\left(\frac{1}{4}\right) - \frac{81}{140}\left(\frac{1}{3}\right) + \frac{52}{105}\left(\frac{1}{2}\right) - \frac{81}{280}\left(\frac{2}{3}\right) + \frac{64}{315}\left(\frac{3}{4}\right) + 0(1)\right]
\end{array}\right] = 0$$

3.1.2. Corollary 2

The local truncation error of (12) is assumed $\xi(\tau)$ to be sufficiently differentiable, and expanding Equation (13) by $\xi(\tau)$ using the Taylor series to get,

$$L_{\frac{1}{4}}[\xi(\tau_n);h]=(1.4002\times10^{-08}),\quad L_{\frac{1}{3}}[\xi(\tau_n);h]=(1.3861\times10^{-08}),\quad L_{\frac{1}{2}}[\xi(\tau_n);h]=(-9.1571\times10^{-07}),$$
$$L_{\frac{2}{3}}[\xi(\tau_n);h]=(1.3861\times10^{-08}),\quad L_{\frac{3}{4}}[\xi(\tau_n);h]=(1.4002\times10^{-08}),\quad L_{1}[\xi(\tau_n);h]=(1.3861\times10^{-08})$$

Proof

Expand (15) using Corollary 2 and collect the like terms to the power of h gives,

$$L_{\frac{1}{4}}[\xi(\tau_n);h]=(1.4002\times10^{-08})C_{07}h^{06}\xi^{06}(\tau_n)+0(h^{08})$$
$$L_{\frac{1}{3}}[\xi(\tau_n);h]=(1.3861\times10^{-08})C_{07}h^{06}\xi^{06}(\tau_n)+0(h^{08})$$
$$L_{\frac{1}{2}}[\xi(\tau_n);h]=(-9.1571\times10^{-07})C_{07}h^{06}\xi^{06}(\tau_n)+0(h^{08})$$
$$L_{\frac{2}{3}}[\xi(\tau_n);h]=(1.3861\times10^{-08})C_{07}h^{06}\xi^{06}(\tau_n)+0(h^{08})$$
$$L_{\frac{3}{4}}[\xi(\tau_n);h]=(1.4002\times10^{-08})C_{07}h^{06}\xi^{06}(\tau_n)+0(h^{08})$$
$$L_{1}[\xi(\tau_n);h]=(1.3861\times10^{-08})C_{07}h^{06}\xi^{06}(\tau_n)+0(h^{08})$$

3.2. Consistency

Equation (12) is consistent as cited in [2], since it has order $p \geq 1$.

3.3. Zero-Stability

To determine the zero-stability of (12), we employ the roots $z_s,\ s=1,2,\dots,n$ of the first characteristic polynomial $\overline{\rho}(z)$, defined by

$$\overline{\rho}(z)=\det\left[zA^{(0)}-E\right] \tag{16}$$

Satisfies $|z_s|\leq 1$ and every root $|z_s|=1$ has multiplicity that does not exceed the order of the differential equation $h\to 0$. Moreover as $h\to 0$, $\rho(z)=z^{r-\mu}(z-1)^{\mu}$, where μ is the order of the differential equation, r is the order of the matrices $A^{(0)}$ *and* E, given by

$$\rho(z)=\left|z\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\right|=\left|\begin{bmatrix}z&0&0&0&-1\\0&z&0&0&-1\\0&0&z&0&-1\\0&0&0&z&-1\\0&0&0&0&z-1\end{bmatrix}\right|=z^4(z-1)$$

Thus, solving for z in

$$z^4(z-1) \tag{17}$$

Gives $z = 0, 0, 0, 0, 1$. Hence, (12) is zero-stable [2].

3.4. Convergence

The (12) is convergent since it is consistent and zero-stable [2].

3.5. Region of Absolute Stability

According to [2], the Boundary Locus Method were used on (12) to obtain the stability polynomial as,

$$\bar{h}(w) = \left(-\frac{1}{241920}w^5 - \frac{1}{241920}w^6\right)h^{12} + \left(-\frac{11}{103680}w^5 - \frac{11}{103680}w^6\right)h^{10} + \left(-\frac{19}{10368}w^5 + \frac{7}{4320}w^6\right)h^8$$
$$+\left(-\frac{29}{1728}w^5 - \frac{29}{1728}w^6\right)h^6 + \left(-\frac{65}{432}w^4 + \frac{101}{864}w^5\right)h^4 + \left(-\frac{1}{2}w^5 - \frac{1}{2}w^6\right)h^2 - 2w^4 + w^5 \tag{18}$$

Using the stability polynomial (18), the region of absolute stability of (12) is shown in Figure 1 as,

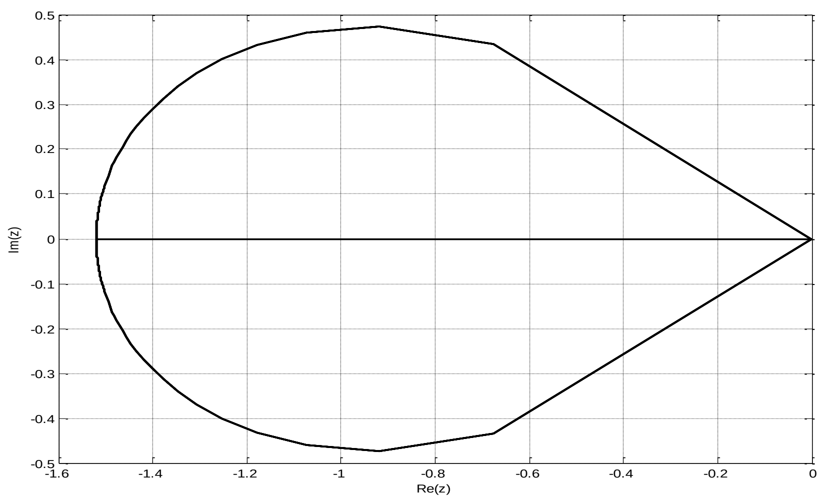

Fig. 1 Stability region

The stability region obtained in Figure 1 is A- stable.

4. Numerical Results

To validate the efficiency and accuracy of the newly developed method, two well-established second-order oscillatory differential equations, the Bettis and Stiefel models, are employed as numerical test cases. These examples are chosen due to their relevance in modeling systems with slowly varying oscillations and their sensitivity to numerical errors over long intervals.

4.1. Example 1: Examine the Second-Order Bettis Oscillatory Differential Equation

$$\xi''(\tau) + \xi'(\tau) = 0.001\cos(\tau),\ \xi(\tau) = 1,\ \xi'(\tau) = 0 \tag{19}$$

With the exact solution of (19) as,

$$\xi(\tau)=\cos(\tau)+0.0005\tau\sin(\tau) \quad (20)$$

Source [1, 2].

4.2. *Example 2: Examine the Second-Order Bettis Oscillatory Differential Equation*

$$\xi''(\tau)+\xi'(\tau)=0.001\sin(\tau),\ \xi(\tau)=0,\ \xi'(\tau)=0.9995 \quad (21)$$

With the exact solution of (21) as

$$\xi(\tau)=\sin(\tau)-0.0005\tau\cos(\tau) \quad (22)$$

Source [1, 2]

The following acronyms were used in the tables and figures.

Acronym	Meaning
τ	Point of evaluation
ESG	Exact Solution Given
ENM	Error in New Method
CSNM	Computed Solution in New Method
EOM16	Error in [1]
ELe21	Error in [2]

Table 1. Comparison of errors in the new method for Example 1 with those of [1, 2]

τ	ESG	CSNM	ENM	EOM16	ELe21
0.1	0.09978366643856425102	0.09978366643856423962	1.1400(-17)	1.0170(-12)	1.2567(-12)
0.2	0.19857132413727709130	0.19857132413727706870	2.2600(-17)	1.4285(-11)	2.1140(-12)
0.3	0.29537690618797073421	0.29537690618797070085	3.3360(-17)	4.9557(-11)	2.3764(-12)
0.4	0.38923413010984991465	0.38923413010984987118	4.3470(-17)	1.0161(-10)	3.4242(-12)
0.5	0.47920614296373040709	0.47920614296373035428	5.2810(-17)	1.7416(-10)	3.3944(-12)
0.6	0.56439487271056245371	0.56439487271056239258	6.1130(-17)	2.6425(-10)	3.3436(-12)
0.7	0.64394999247214148272	0.64394999247214141446	6.8260(-17)	3.7579(-10)	4.2949(-12)
0.8	0.71707740821578389546	0.71707740821578382138	7.4080(-17)	5.0602(-10)	4.2574(-12)
0.9	0.78304718514176158945	0.78304718514176151102	7.8430(-17)	6.5904(-10)	5.2344(-12)
1.0	0.84120083365496243679	0.84120083365496235556	8.1230(-17)	8.3225(-10)	6.2265(-12)

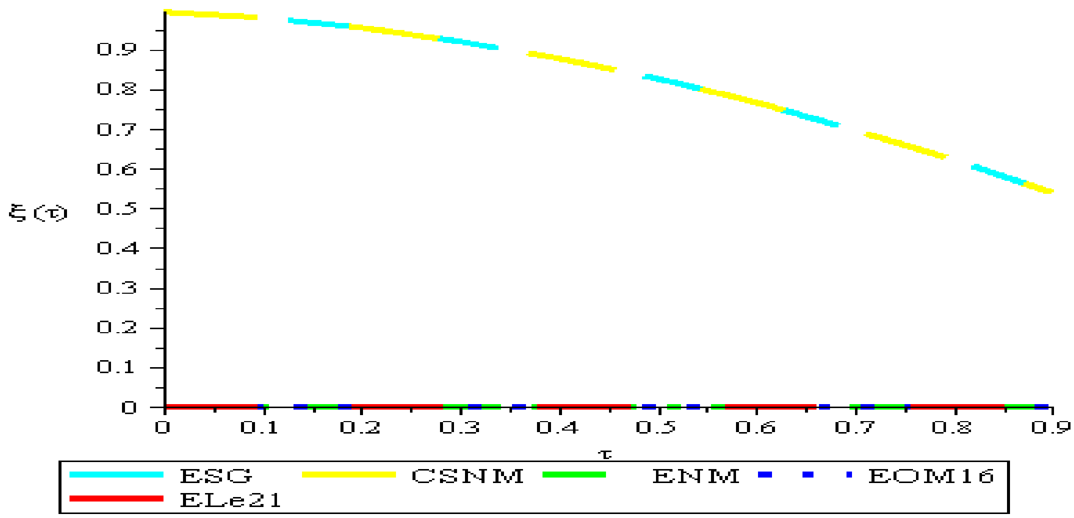

Fig. 2 Graphical representational curve of table 1

Table 2. Comparison of errors in the new method for Example 2 with those of [1, 2]

τ	ESG	CSNM	ENM	EOM16	ELe21
0.1	0.99500915694885810751	0.99500915694885810801	5.0000(-19)	1.0169e-11	2.8269(-12)
0.2	0.98008644477432113724	0.98008644477432113925	2.0100(-18)	2.0390e-11	5.8994(-12)
0.3	0.95538081715660522058	0.95538081715660522503	4.4500(-18)	1.5451e-13	6.8309(-12)
0.4	0.92113887767134681290	0.92113887767134682070	7.8000(-18)	8.1063e-11	1.4991(-12)
0.5	0.87770241827502376687	0.87770241827502377886	1.1990(-17)	2.5377e-10	1.8395(-12)
0.6	0.82550500765169680785	0.82550500765169682480	1.6950(-17)	5.4848e-10	1.6559(-12)
0.7	0.76506766347502161813	0.76506766347502164069	2.2560(-17)	9.9571e-10	1.2970(-11)
0.8	0.69699365178352523002	0.69699365178352525873	2.8710(-17)	1.6260e-10	8.4312(-11)
0.9	0.62196246537999682400	0.62196246537999685928	3.5280)-17)	2.4697e-10	5.3240(-11)
1.0	0.54072304136054366565	0.54072304136054370774	4.2090(-17)	3.5575e-10	3.2126(-11)

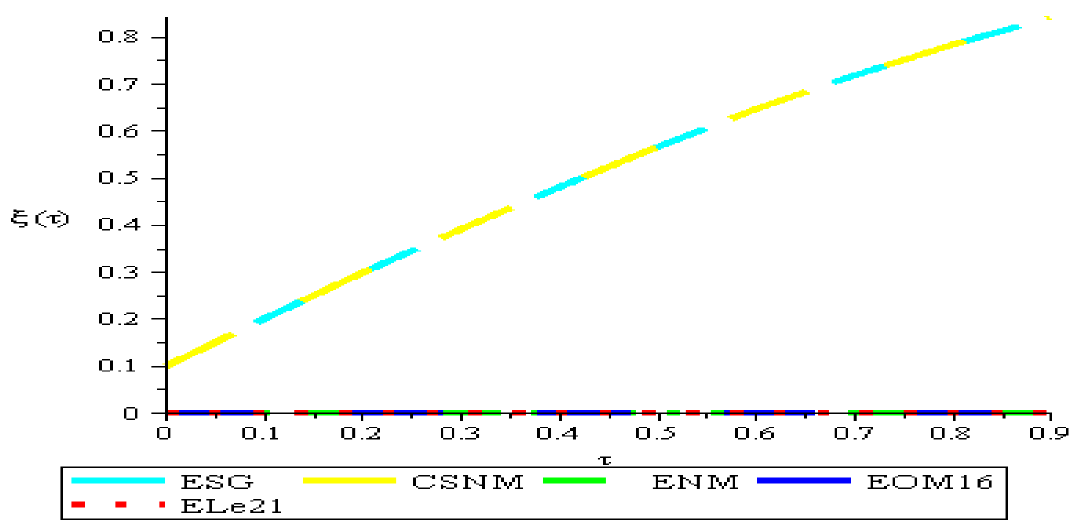

Fig. 3 Graphical representational curve of table 2

5. Discussion of Results

The numerical test of the new method demonstrates its numerical robustness since it confirms its capability to solve second-order oscillatory differential equations. The method possesses a balanced order of accuracy, whose error constant is determined by carrying out a Taylor series expansion of the corresponding linear operator. It satisfies the consistency requirement since it possesses a high enough order as suggested in previous research.

Furthermore, the method is zero-stable, and its roots for the characteristic polynomial meet the given conditions such that errors do not grow exponentially with further steps. Because of consistency and zero-stability, the method is convergent, guaranteeing that the numerical solution converges towards the exact solution as the step size tends to be small. Finally, the boundary locus technique illustrates that the method possesses an unambiguous region of absolute stability as revealed in Figure 1, and hence can be relied upon to solve stiff or highly oscillatory problems numerically without instability.

In Example 1, solving the Bettis Oscillatory Differential Equation, the comparison given in Table 1 shows a very close similarity between the exact solution and the new method solution. At all points of evaluation, the new method consistently replicates the exact values, showing its credibility and accuracy in the prediction of second-order oscillatory systems' behavior. Compared to the previous methods proposed in [1, 2], the new method demonstrates improved performance and closer approximation to the solution.

The graphical plot in Figure 2 also validates the correctness of the new method. The curve of the new method overlaps closely with the curve of the exact solution, and there is almost negligible visual difference. This shows the high accuracy of the method, especially in maintaining the amplitude and phase characteristics of the oscillatory nature of the Bettis system. The trend is the same in the graph as the numerical one experienced in the table.

Example 2 presents the Stiefel Oscillatory Differential Equation, and a comparison is shown in Table 2 between exact results and results from the new method. Like the first example, results from the application of the new technique are almost exact to the exact solutions, showing the capability of the method to solve more complex oscillatory systems. This demonstrates the robustness of the method, even in stiff or high-dimensional systems where other methods are ineffective. Figure 3 demonstrates visually the correctness validation of Example 2.

The new method's output curve still follows in close proximity to the exact solution at every point of calculation, emphasizing how effective it is at delivering solution accuracy. This consistency confirms that the method is apt for solving second-order oscillatory differential equations, especially those encountered in applications demanding high numerical accuracy and stability, like those represented through the Stiefel formulation.

6. Conclusion

The article proposes a novel continuous hybrid block method for the solution of Bettis and Stiefel second-order oscillatory differential equations, in which slowly varying amplitude and phase occur. Unlike traditional methods, which recast such equations into first-order systems at the cost of computational efficiency and accuracy, the new method solves the second-order form directly with power series polynomials as a basis and calculates multiple points in parallel. Systematic analysis checks the method's favorable numerical properties, including consistency, stability, and convergence. Numerical computations validate its enhanced accuracy and efficiency compared to existing methods, and therefore, it comes out as a robust, reliable tool for scientific and engineering problems with oscillatory dynamics.

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