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Original Article

Fourth Derivative Methods for Solving Fourth-Order Initial Value Problems Using an Optimized Three-Step, Two-Off Grid

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Abstract - In order to solve fourth-order ordinary differential equations, this article presents an optimized three-step, two-off-grid hybrid point fourth derivative method. The method uses an exponential function as the basis function for a chosen three hybrid points, appropriately optimizing one of the two off-grid points by setting the principal term of the local truncation error to zero and using the local truncation error to determine the approximate values of the unknown parameter. The method's basic properties were examined, and the developed method was applied to work out some fourth order initial value problems of ordinary differential equations. Based on the numerical results, it is observed that our new approach provides a better approximation than the existing method when compared to our result.

Keywords - Optimize off-grid, Local truncation error, Free-parameter, Three-step, Fourth derivative.

1. Introduction

In this manuscript, we reflect on fourth order three-step one, optimized hybrid point approach form

$$\varphi^{(iv)} = \gamma\left(\mathrm{t}, \varphi(t), \varphi'(t), \varphi''(t), \varphi'''(t)\right),\ \varphi\left(t_0\right) = \varphi_0, \varphi'\left(t_0\right) = \varphi'_0, \varphi''\left(t_0\right) = \varphi''_0, \varphi'''\left(t_0\right) = \varphi'''_0 \quad (1)$$

An essential mathematical tool for modeling a range of physical processes that occur in several scientific and engineering fields is the fourth order ordinary differential equation, including fluid dynamics, vibration analysis, control systems, and structural mechanics. According to [1], the fourth derivative can be optimized to predict, improve, and enhance computational efficiency and solution accuracy with superior accuracy and a good region of absolute stability. Equation (1) is used to solve problems in a number of real-world domains that frequently occur in physical systems in science and engineering, such as mechanics, control theory, beam deflection, and ship dynamics, among others. However, the majority of these physical issues are complex systems for which an analytical solution is exceedingly challenging, if not possible. Numerical techniques, which solve (1), are therefore essential tools. As a result, fourth order ordinary differential equations have garnered a lot of attention from researchers, and as a result, theoretical and numerical studies addressing (1) have recently surfaced in literature. As demonstrated by Kuboye et al. (2020) and Kayode and Adeyeye (2011), the well-known and accepted method of solving (1) is to reduce it to a system of differential equations of the first order, which subsequently results in a computational burden. For solving (1), a variety of approaches have been put forth, including hybrid approaches, predictor-corrector approaches, and recently refined approaches. Among those who have recently embraced the hybrid approach as an alternative to the direct method for approximating (1) are Adesanya et al. (2013), Adebayo and Adebola (2016), and Adoghe and Omole (2019). Akinnukawe et al. (2024) have proposed an optimized approach by using a novel fourth-order block algorithm to numerically solve fourth-order initial value problems; Bothayna and Sadeemr (2019) have proposed an optimization of a two-step block method with three hybrid points for solving third-order initial value problems;

Kashkari and Syam (2019) have proposed an optimization of a one-step block method with three hybrid points to solve first-order ordinary differential equations; Singh et al. (2019) have also proposed an adaptive optimized step size hybrid method for integrating differential systems; and Joshua (2022) has used the optimized approach to derive a two-step second derivative method for solving stiff systems. The three-step fourth derivative method created for this study optimizes one of the two off-grid placements by setting the principal term of the local truncation error to zero and using the local truncation error to estimate the values of the unknown parameter.

The structure of the paper is as follows: The techniques and resources used to build the method are covered in section 2. Section 4 provides the numerical experiments where the effectiveness of the derived method is tested on some stiff problems; the results were discussed, and a conclusion was made in the section. Section 3 examines the analysis of the method's fundamental properties, such as convergence and stability region.

2. Derivation of the Method

The primary method, which will be configured to obtain the block method, is generated using the three-step process. Using the form's exponentially fitted approximation solution, it provides,

$$\varphi(t)=\varsigma_0(t)\varphi_n+\varsigma_1(t)\varphi'_n+\varsigma_2(t)\varphi''_n+\varsigma_3(t)\varphi'''_n+h^4\left[\sum_{j=0}^{3}\varsigma_j(t)e^{t^j}\gamma_{n+j}+\varsigma_k(t)e^{t^k}\gamma_{n+k}\right], k=\mu_1,\mu_2 \tag{2}$$

$\varsigma_i(t)$ are polynomials $\varphi_{n+j}=\varphi\left(t_{n+j}\right)$, $\gamma_{n+j}=\gamma\left(t_{n+j},\varphi_{n+j}\right)$ $\xi=\frac{t-t_{n+4}}{h}$

Equation (2) is obtained by taking into consideration the exponentially-fitted approximate solution of the form,

$$\varphi(x)=\sum_{j=0}^{s+r-1}\varsigma_j e^{\xi^j} \tag{3}$$

Where $r=4$ and $s=6$ are The solution to (1) is thought to be the records of the collocation points and interpolation points, respectively. The following requirements are enforced;

$$\left.\begin{aligned}
&\varphi_{n+j} = \varphi\left(t_{n+j}\right),\ j=0\\
&\varphi'_{n+j} = \varphi'\left(t_{n+j}\right),\ j=0\\
&\varphi''_{n+j} = \varphi''\left(t_{n+j}\right),\ j=0\\
&\varphi'''\left(t_{n+j}\right) = \varphi''\left(t_{n+j}\right),\ j=0\\
&\varphi^{iv}\left(t_{n+j}\right) = \gamma_{n+j}\ ,\ j=0,\mu_1,\mu_2,1,2,3
\end{aligned}\right\} \tag{4}$$

By differentiating equation (3) thrice gives,

$$\varphi'(t) = \sum_{j=0}^{r} jt^{j-1}\varsigma_j(t)e^{t^j} = \gamma(t,\varphi)$$

$$\varphi''(t) = \sum_{j=0}^{r} jt^{j-2}\varsigma_j(t)e^{t^j}\left(j+jt^j-1\right) = \gamma(t,\varphi,\varphi')$$

$$\varphi'''(t) = \sum_{j=0}^{r} jt^{j-3}\varsigma_j(t)e^{t^j}\left(j^2t^{2j}-3j-3jt^j+3j^2t^j+j^2+2\right) = \gamma(t,\varphi,\varphi',\varphi'') \tag{5}$$

Putting (5) into (1) gives,

$$^{j}\varphi''''(t) = \gamma\left(t,{}^{j}\varphi,{}^{j}\varphi',{}^{j}\varphi'',{}^{j}\varphi'''\right) = \varsigma_0\varphi + h\varsigma_1\varphi' + h^2\varsigma_2\varphi'' + h^3\varsigma_3\varphi''' + \sum_{i=4}^{9} i\,(i-1)(i-2)(i-3)\,\varsigma_i t^{i-4}\ ,\ j=1,\ldots,4 \tag{6}$$

Collocating (6) at all points $t_{n+s}, s=0,\mu_1,\mu_2,1,2,3$ and Interpolating (3) and (5) $t_{n+r}, r=0$ give a structure of a nonlinear equation of the form,

$$\begin{bmatrix}
6 & 6t_n & 3t_n^2 & t_n^3 & \frac{1}{4}t_n^4 & \frac{1}{20}t_n^5 & \frac{1}{120}t_n^6 & \frac{1}{840}t_n^7 & \frac{1}{6720}t_n^8 & \frac{1}{60480}t_n^9\\
0 & 6 & 6t_n & 3t_n^2 & t_n^3 & \frac{1}{4}t_n^4 & \frac{1}{20}t_n^5 & \frac{1}{120}t_n^6 & \frac{1}{840}t_n^7 & \frac{1}{6720}t_n^8\\
0 & 0 & 6 & 6t_n & 3t_n^2 & t_n^3 & \frac{1}{4}t_n^4 & \frac{1}{20}t_n^5 & \frac{1}{120}t_n^6 & \frac{1}{840}t_n^7\\
0 & 0 & 0 & 6 & 6t_n & 3t_n^2 & t_n^3 & \frac{1}{4}t_n^4 & \frac{1}{20}t_n^5 & \frac{1}{120}t_n^6\\
0 & 0 & 0 & 0 & 6 & 6t_n & 3t_n^2 & t_n^3 & \frac{1}{4}t_n^4 & \frac{1}{20}t_n^5\\
0 & 0 & 0 & 0 & 6 & 6t_n & 3t_{n+\mu_1}^2 & t_{n+\mu_1}^3 & \frac{1}{4}t_{n+\mu_1}^4 & \frac{1}{20}t_{n+\mu_1}^5\\
0 & 0 & 0 & 0 & 6 & 6t_n & 3t_{n+\mu_2}^2 & t_{n+\mu_2}^3 & \frac{1}{4}t_{n+\mu_2}^4 & \frac{1}{20}t_{n+\mu_2}^5\\
0 & 0 & 0 & 0 & 6 & 6t_n & 3t_{n+1}^2 & t_{n+1}^3 & \frac{1}{4}t_{n+1}^4 & \frac{1}{20}t_{n+1}^5\\
0 & 0 & 0 & 0 & 6 & 6t_n & 3t_{n+2}^2 & t_{n+2}^3 & \frac{1}{4}t_{n+2}^4 & \frac{1}{20}t_{n+2}^5\\
0 & 0 & 0 & 0 & 6 & 6t_n & 3t_{n+3}^2 & t_{n+3}^3 & \frac{1}{4}t_{n+3}^4 & \frac{1}{20}t_{n+3}^5
\end{bmatrix}^{-1}
\begin{bmatrix}\delta_0\\ \delta_1(m)\\ \delta_2(m)\\ \delta_3(m)\\ \delta_4(m)\\ \delta_5(m)\\ \delta_6(m)\\ \delta_7(m)\\ \delta_8(m)\\ \delta_9(m)\end{bmatrix}
=
\begin{bmatrix}\varphi_n\\ \varphi'_n\\ \varphi''_n\\ \varphi'''_n\\ \gamma_n\\ \gamma_{n+\mu_1}\\ \gamma_{n+\mu_2}\\ \gamma_{n+1}\\ \gamma_{n+2}\\ \gamma_{n+3}\end{bmatrix} \tag{7}$$

Solving (7) for $\delta_i{}'s$ by the Gaussian elimination method to obtain the unknown values in terms of the parameter u,v,w provides a continuous multistep hybrid linear approach of the following form;

$$p(t) = \varsigma_0(t)\varphi_n + \varsigma_1(t)\varphi'_n + \varsigma_2(t)\varphi''_n + \varsigma_3(t)\varphi'''_n + h^4\left[\sum_{j=0}^{3}\varsigma_j(t)e^{t^j}\gamma_{n+j} + \varsigma_k(t)e^{t^k}\gamma_{n+k}\right], k=\mu_1,\mu_2 \tag{8}$$

Where, $\varsigma_0 = 1 \quad \varsigma_1 = m \quad \varsigma_2 = \frac{1}{2}m^2 \quad \varsigma_3 = \frac{1}{6}m^3$

$$\varsigma_0 = -\frac{1}{90720}m^4\left(\frac{\begin{array}{c}-462\mu_1m^2+108\mu_1m^3-462m^2\mu_2-9m^4\mu_2+108\mu_2m^3-9m^4\mu_1+756m\mu_2-3780\mu_1\mu_2\\ -252m^2+198m^3-54m^4+5m^5-252m^2\mu_1\mu_2+18m^3\mu_1\mu_2+1386m\mu_1\mu_2\end{array}}{\mu_1\mu_2}\right)$$

$$\varsigma_1 = \frac{1}{15120}m^5\left(\frac{-252m+756\mu_2+108m^2\mu_2-9m^3\mu_2-462m\mu_2+198m^2-54m^3+5m^4}{\mu_1(\mu_1-3)(\mu_1-1)(\mu_1-2)(\mu_1-\mu_2)}\right)$$

$$\varsigma_2 = -\frac{1}{15120}m^5\left(\frac{-252m+756\mu_2+108m^2\mu_2-9m^3\mu_2-462m\mu_2+198m^2-54m^3+5m^4}{\mu_2(\mu_2-3)(\mu_2-1)(\mu_2-2)(\mu_1-\mu_2)}\right)$$

$$\varsigma_3 = \frac{1}{30240}m^5\left(\frac{\begin{array}{c}90m^2\mu_1-9m^3\mu_1+90m^2\mu_2-9m^3\mu_2-252m\mu_1-252m\mu_2\\ +756\mu_1\mu_2+108m^2-45m^3+5m^4+18m^2\mu_1\mu_2-210m\mu_1\mu_2\end{array}}{(\mu_2-1)(\mu_1-1)}\right)$$

$$\varsigma_4 = \frac{1}{30240}m^5\left(\frac{\begin{array}{c}72m^2\mu_1-9m^3\mu_1+72m^2\mu_2-9m^3\mu_2-126m\mu_1-126m\mu_2+378\mu_1\mu_2\\ +54m^2-36m^3+5m^4+18m^2\mu_1\mu_2-168m\mu_1\mu_2\end{array}}{(\mu_2-2)(\mu_1-2)}\right)$$

$$\varsigma_5 = \frac{1}{90720}m^5\left(\frac{\begin{array}{c}54m^2\mu_1-9m^3\mu_1+54m^2\mu_2-9m^3\mu_2-84m\mu_1-84m\mu_2+252\mu_1\mu_2\\ +36m^2-27m^3+54m^4+18m^2\mu_1\mu_2-126m\mu_2\mu_1\end{array}}{(\mu_2-3)(\mu_1-3)}\right)$$

By substituting $m=1$ in (8), a multistep formula to approximate the solution of (1.1) at the point t_{n+1} yields,

$$t_{n+1} = \varphi_n + h\varphi'_n + \frac{1}{2}\varphi''_n h^2 + \frac{1}{6}h^3\varphi'''_n + h^4\left[\begin{array}{l}\left(\frac{1}{90720}\frac{(-393r-393s+2628rs+103)}{rs}\right)g_n + \left(\frac{1}{15120}\frac{(393s-103)}{r(r-3)(r-1)(r-2)(r-s)}\right)g_{n+r}\\ -\left(\frac{1}{15120}\frac{(393s-103)}{s(s-3)(s-1)(s-2)(r-s)}\right)g_{n+s} + \left(\frac{1}{30240}\frac{(-171r-171s+564rs+68)}{(s-1)(r-1)}\right)g_{n+1}\\ -\left(\frac{1}{30240}\frac{(-63r-63s+228rs+23)}{(s-2)(r-2)}\right)g_{n+2} + \left(\frac{1}{90720}\frac{(-39r-39s+144rs+14)}{(s-3)(r-3)}\right)g_{n+3}\end{array}\right] \tag{9}$$

Also, by substituting $m=1$ in the first derivative of (8), a multistep formula to approximate the solution of (1) at the point t'_{n+1} yields

$$ht'_{n+1}=h\varphi'_n+\varphi''_n h^2+\tfrac{1}{2}h^3\varphi'''_n+h^4\left[\begin{array}{l}\left(\frac{1}{10080}\frac{(-188r-188s+1064rs+57)}{rs}\right)g_n+\left(\frac{1}{1680}\frac{(188s-57)}{r(r-3)(r-1)(r-2)(r-s)}\right)g_{n+r}\\-\left(\frac{1}{1680}\frac{(188r-57)}{s(s-3)(s-1)(s-2)(r-s)}\right)g_{n+s}+\left(\frac{1}{3360}\frac{(-106r-106s+294rs+49)}{(s-1)(r-1)}\right)g_{n+1}\\-\left(\frac{1}{3360}\frac{(-36r-36s+112rs+15)}{(s-2)(r-2)}\right)g_{n+2}+\left(\frac{1}{10080}\frac{(-22r-22s+70rs+9)}{(s-3)(r-3)}\right)g_{n+3}\end{array}\right] \quad (10)$$

Also, by substituting $m=1$ in the second derivative of (8), a multistep formula to approximate the solution of (1) at the point t''_{n+1} yields,

$$h^2t''_{n+1}=\varphi''_n h^2+h^3\varphi'''_n+h^4\left[\begin{array}{l}\left(\frac{1}{2520}\frac{(-147r-147s+679rs+53)}{rs}\right)g_n+\left(\frac{1}{420}\frac{(147s-53)}{r(r-3)(r-1)(r-2)(r-s)}\right)g_{n+r}\\-\left(\frac{1}{420}\frac{(147r-53)}{s(s-3)(s-1)(s-2)(r-s)}\right)g_{n+s}+\left(\frac{1}{840}\frac{(-119r-119s+266rs+66)}{(s-1)(r-1)}\right)g_{n+1}\\-\left(\frac{1}{840}\frac{(-35r-35s+91rs+17)}{(s-2)(r-2)}\right)g_{n+2}+\left(\frac{1}{2520}\frac{(-21r-21s+56rs+10)}{(s-3)(r-3)}\right)g_{n+3}\end{array}\right] \quad (11)$$

Finally, by substituting $m=1$ in the third derivative of (8), a multistep formula to approximate the solution of (1) at the point t'''_{n+1} yields,

$$h^3t'''_{n+1}=\varphi'''_n h^3+h^4\left[\begin{array}{l}\left(\frac{1}{360}\frac{(-38r-38s+135rs+17)}{rs}\right)g_n+\left(\frac{1}{60}\frac{(38s-17)}{r(r-3)(r-1)(r-2)(r-s)}\right)g_{n+r}\\-\left(\frac{1}{60}\frac{(38s-17)}{s(s-3)(s-1)(s-2)(r-s)}\right)g_{n+s}+\left(\frac{1}{120}\frac{(-57r-57s+95rs+40)}{(s-1)(r-1)}\right)g_{n+1}\\-\left(\frac{1}{120}\frac{(-12r-12s+25rs+7)}{(s-2)(r-2)}\right)g_{n+2}+\left(\frac{1}{360}\frac{(-7r-7s+15rs+4)}{(s-3)(r-3)}\right)g_{n+3}\end{array}\right] \quad (12)$$

2.1. Algorithm for the Derivation of the Optimize Method

The algorithm for the implementation of the optimize three-step method is given by the following steps;

Step 1 : Expanding (12) using the taylor series to obtain the corresponding Local truncation error given as

$$L\left[y(x);h\right]=\frac{-1}{50400}\left(-119\mu_1-119\mu_2+266\mu_1\mu_2+66\right)+o(h^{10}) \quad (13)$$

Step 2 : Equate the principal term of the local truncation errors in (13) to zero, hence keeping μ_2 as a free parameter by assigning $\mu_2=\frac{2}{3}$ in (13) to obtain the values $\mu_1=\frac{8}{35}$

Step 3 : Substitute the values of μ_1 and μ_2 as obtain in step 2 into (9) – (12), gives the equations as shown in Tables 1 to 4 below.

Table 1. Coefficients of $\varsigma's_i$ (8) 's first derivative, which is assessed at every point

	φ_n	$h\varphi'_n$	$h^2\varphi''_n$	$h^3\varphi'''_n$	$h^4\gamma_n$	$h^4\gamma_{n+\frac{8}{35}}$	$h^4\gamma_{n+\frac{2}{3}}$	$h^4\gamma_{n+1}$	$h^4\gamma_{n+2}$	$h^4\gamma_{n+3}$
$\varphi_{n+\frac{8}{35}}$	1	$\frac{8}{35}$	$\frac{32}{1225}$	$\frac{256}{128625}$	$\frac{56922990592}{709340748046875}$	$\frac{2493219328}{58845939429375}$	$\frac{-179400801792}{12689317826171875}$	$\frac{33649983488}{6384066732421875}$	$\frac{-6105325568}{2198956318945\text{[illegible]}}$	$\frac{3579969536}{\text{[illegible]}47455974609375}$
$\varphi_{n+\frac{2}{3}}$	1	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{4}{81}$	$\frac{58568}{18600435}$	$\frac{1776740000}{330794919009}$	$-\frac{1258}{2464749}$	$\frac{39016}{167403915}$	$-\frac{2528}{192204495}$	$\frac{13528}{1262965365}$
φ_{n+1}	1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1769}{161280}$	$\frac{79533125}{2868244992}$	$\frac{4149}{1442560}$	$\frac{1}{9072}$	$-\frac{47}{2499840}$	$\frac{1}{570360}$
φ_{n+2}	1	2	2	$\frac{4}{3}$	$\frac{76}{945}$	$\frac{6002500}{16806123}$	$\frac{792}{5635}$	$\frac{104}{1215}$	$\frac{2}{651}$	$-\frac{88}{641655}$
φ_{n+3}	1	3	$\frac{9}{2}$	$\frac{9}{2}$	$\frac{3051}{17920}$	$\frac{58524375}{35410432}$	$\frac{334611}{1442560}$	$\frac{321}{280}$	$\frac{48519}{277760}$	$-\frac{81}{76048}$

Table 2. Coefficients of $\varsigma's_i$ (8) 's first derivative, which is assessed at every point

	φ_n	φ'_n	$h\varphi''_n$	$h^2\varphi'''_n$	$h^3\gamma_n$	$h^3\gamma_{n+\frac{8}{35}}$	$h^3\gamma_{n+\frac{2}{3}}$	$h^3\gamma_{n+1}$	$h^3\gamma_{n+2}$	$h^3\gamma_{n+3}$
$\varphi_{n+\frac{8}{35}}$	0	1	$\frac{8}{35}$	$\frac{32}{1225}$	$\frac{2891211968}{2251875390625}$	$\frac{1476721984}{1681312555125}$	$\frac{-97193820672}{18432062528}$	$\frac{18423062528}{182401906640625}$	$\frac{-1107923456}{209424411328125}$	$\frac{1945542656}{4587070170703125}$
$\varphi_{n+\frac{2}{3}}$	0	1	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{6857}{459270}$	$\frac{277615625}{8167775778}$	$-\frac{19}{182574}$	$\frac{1256}{2066715}$	$-\frac{209}{4745790}$	$\frac{584}{155922165}$
φ_{n+1}	0	1	1	$\frac{1}{2}$	$\frac{593}{17920}$	$\frac{307628125}{2868244992}$	$\frac{39771}{1442560}$	$-\frac{23}{18144}$	$\frac{17}{833280}$	$-\frac{1}{2281440}$
φ_{n+2}	0	1	2	2	$\frac{61}{630}$	$\frac{7503125}{11204082}$	$\frac{2349}{11270}$	$\frac{968}{2835}$	$\frac{23}{1302}$	$-\frac{152}{213885}$
φ_{n+3}	0	1	3	$\frac{9}{2}$	$\frac{2241}{17920}$	$\frac{67528125}{35410432}$	$\frac{177147}{1442560}$	$\frac{2061}{1120}$	$\frac{138267}{277760}$	$\frac{1089}{152096}$

Table 3. Coefficients of $\varsigma' s_i$ considering (8) 's second derivative, which is assessed at every point

	φ_n	φ'_n	φ''_n	$h\varphi'''_n$	$h^2\gamma_n$	$h^2\gamma_{n+\frac{8}{35}}$	$h^2\gamma_{n+\frac{2}{3}}$	$h^2\gamma_{n+1}$	$h^2\gamma_{n+2}$	$h^2\gamma_{n+3}$
$\varphi_{n+\frac{8}{35}}$	0	0	1	$\frac{8}{35}$	$\frac{8351129408}{579053671875}$	$\frac{8390304}{593055575}$	$-\frac{39328841088}{10358626796875}$	$\frac{273565696}{193017890625}$	$-\frac{440622464}{5983554609375}$	$\frac{2315487232}{393177443203125}$
$\varphi_{n+\frac{2}{3}}$	0	0	1	$\frac{2}{3}$	$\frac{5891}{131220}$	$\frac{2528553125}{16335551556}$	$\frac{16}{621}$	$-\frac{968}{295245}$	$\frac{19}{338985}$	$-\frac{8}{3182085}$
φ_{n+1}	0	0	1	1	$\frac{373}{5760}$	$\frac{7503125}{26557824}$	$\frac{7857}{51520}$	0	$\frac{7}{29760}$	$-\frac{1}{61110}$
φ_{n+2}	0	0	1	2	$\frac{1}{180}$	$\frac{7503125}{7469388}$	$-\frac{81}{805}$	$\frac{136}{135}$	$\frac{8}{93}$	$-\frac{88}{30555}$
φ_{n+3}	0	0	1	3	$\frac{183}{640}$	$\frac{7503125}{8852608}$	$\frac{6561}{7360}$	$\frac{13}{10}$	$\frac{11043}{9920}$	$\frac{6}{97}$

Table 4. Coefficients of $\varsigma' s_i$ Considering (8) 's third derivative, which is assessed at every point

	φ_n	φ'_n	φ''_n	φ'''_n	$h\gamma_n$	$h\gamma_{n+\frac{8}{35}}$	$h\gamma_{n+\frac{2}{3}}$	$h\gamma_{n+1}$	$h\gamma_{n+2}$	$h\gamma_{n+3}$
$\varphi_{n+\frac{8}{35}}$	0	0	0	1	$\frac{209807188}{2363484375}$	$\frac{31456708}{196071435}$	$-\frac{1343171808}{42280109375}$	$\frac{248586752}{21271359375}$	$-\frac{14644064}{24422671875}$	$\frac{76662272}{1604805890625}$
$\varphi_{n+\frac{2}{3}}$	0	0	0	1	$\frac{4771}{87480}$	$\frac{4359315625}{10890367704}$	$\frac{7141}{28980}$	$-\frac{3512}{98415}$	$\frac{541}{451980}$	$-\frac{632}{7424865}$
φ_{n+1}	0	0	0	1	$\frac{373}{5760}$	$\frac{262609375}{717061248}$	$\frac{23571}{51520}$	$\frac{181}{1620}$	$-\frac{7}{29760}$	$\frac{1}{122220}$
φ_{n+2}	0	0	0	1	$-\frac{79}{360}$	$\frac{52521875}{44816328}$	$-\frac{3483}{3220}$	$\frac{728}{405}$	$\frac{631}{1860}$	$-\frac{232}{30555}$
φ_{n+3}	0	0	0	1	$\frac{621}{640}$	$-\frac{52521875}{26557824}$	$\frac{194643}{51520}$	$-\frac{101}{60}$	$\frac{16083}{9920}$	$\frac{3957}{13580}$

(14)

3. Analysis of Basic Properties of the Method

3.1. Order of the Block

Let the linear operator $L\{y(x):h\}$ associated with the discrete block method (14) be defined

$$L\{\varphi(t):h\} = A^{(0)}\varphi_m^i - \sum_{j=0}^{3} h^i e_i \varphi_n^i - h^4\left(g_i\gamma(\varphi_n) + p_i\gamma(\varphi_m)\right) \tag{15}$$

Comparing the coefficient of h after expanding (12) in the Taylor series yields

$$L\{y(x):h\} = C_0 y(x) + C_1 y'(x) + ,\ldots + C_p h^p y^p(x) + C_{p+1}h^{p+1}y^{p+1}(x) + C_{p+2}h^{p+2}y^{p+2}(x) + C_{p+3}h^{p+3}y^{p+3}(x) + \ldots$$

3.1.1. Definition

It is asserted that linear operator L and its corresponding block formula are of order. p, *if* $C_0 = C_1 = ,\ldots = C_p = C_{p+1} = C_{p+2} = C_{p+3} = 0.$ *and* $C_{p+4} \neq 0.$ C_{p+4}. It is referred to as the error constant and suggests that the truncation error can be found using. $t_{n+k} = C_{p+3}h^{p+3}y^{p+3}(x) + O\left(h^{p+4}\right)$.

Our approach involves extending (14) in the Taylor series, then comparing the coefficient of h yields $C_0 = C_1 = C_2 = C_3 = \ldots = C_7 = 0$, and

$$C_{10} = \begin{bmatrix} -\frac{1242821607424}{351921678624755859375}, -\frac{15217}{87887055375}, -\frac{193}{571536000}, \frac{53}{4465125}, \frac{159}{784000}, -\frac{4986438656}{74480778544921875} \\ -\frac{1258}{1953045675}, -\frac{257}{12700800}, -\frac{467}{99225}, -\frac{17131}{156800}, -\frac{5906887936}{6384066732421875}, -\frac{19}{217005075}, \frac{23}{12700800}, \\ \frac{886320180988492}{393186331676988281 25}, -\frac{144842602584717907}{18639944612835000000 0}, -\frac{16780608}{2251875390625}, \frac{16}{1476225}, -\frac{23}{88905600}, \frac{8}{14175}, -\frac{8934287}{2622352320} \end{bmatrix}$$

Hence our method is of order five (5).

3.2. Consistency

The optimized hybrid block method (14) is said to be consistent if it has an order more than or equal to one. Therefore, our method is consistent.

3.3. Zero Stability of Our Method

3.3.1. Definition

A hybrid block approach that optimizes the fourth derivative is considered zero-stable if the roots $z_i, i = \mu_1, \mu_2, 1, 2, 3$ of the first characteristic polynomial $\rho(z) = 0$ that is $\rho(z) = \det\left[\sum_{j=0}^{k} A^{(i)} z^{k-i}\right] = 0$, Satisfies $|z_i| \leq 1$ and for those roots with $z_i = 1$, multiplicity must not exceed two, Hence, our method is zero-stable.

3.4. Consistency

Theorem 3.1 [6]: The necessary and sufficient conditions for the third derivative block methods to be convergent are that they must be consistent and zero-stable. Because it is zero-stable and consistent, the three-step fourth derivative technique with one optimized off grid hybrid point is convergent.

3.5. Linear Stability

The test equation is used to discuss Hairer E. and Wanner G.'s concept of A-stability.

$$y^{(k)} = \lambda^k y \tag{16}$$

To yield,

$$Y_m = \mu(z) Y_{m-1}, z = \lambda h \tag{17}$$

Where, $\mu(z)$ is the amplification matrix given by $\mu(z)$

$$\mu(z) = -\left(\xi^1 - \varsigma^{(1)} - z^4 \eta^{(1)}\right)^{-1} \left(\xi^{(0)} - \varsigma^{(0)} - z^4 \eta^{(0)}\right) \tag{18}$$

The matrix $\mu(z)$ has Eigenvalues $(0, 0, \ldots, \xi_k)$ where ξ_k is called the stability function. Consequently, the three-step optimizer fourth derivative approach with three off-grid hybrid points' stability function is provided by,

$$\xi = \frac{16296z^5 - 176614z^4 + 249366z^3 + 250871z^2 + 27165z - 302400}{384z^5 - 5920z^4 + 42240z^3 - 166560z^2 + 347520z - 302400}$$

3.6. Regions of Absolute Stability

The stability polynomial for optimizing half step with three offstep points gives,

$$-h^5\left(\tfrac{2}{1575}w^5 + \tfrac{97}{1800}w^4\right) + h^4\left(\tfrac{37}{1890}w^5 - \tfrac{2137}{7560}w^4\right) - h^3\left(\tfrac{44}{315}w^5 + \tfrac{53}{63}w^4\right)$$
$$-h^2\left(\tfrac{101}{63}w^4 - \tfrac{347}{630}w^5\right) - h\left(\tfrac{362}{315}w^5 + \tfrac{583}{315}w^4\right) + w^5 - w^4$$

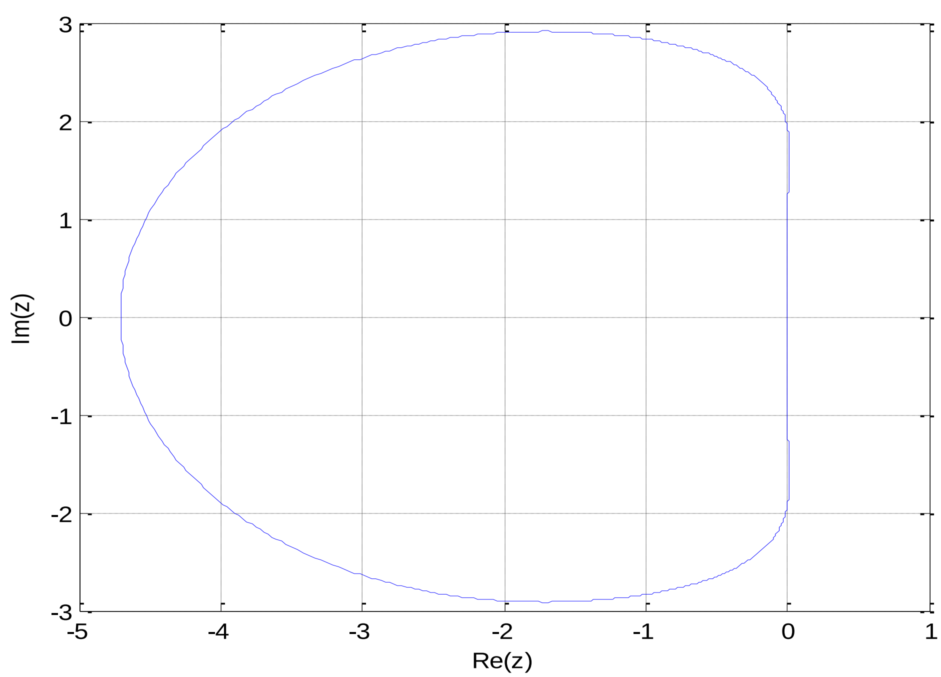

Fig. 1 The new method's region of absolute stability

4. Numerical Result

A few mathematical examples were looked at in order to assess the freshly established procedure's correctness. The outcomes are contrasted with the current approaches put forth by Ukpebor (2020), Victor & Solomon (2021), Akinnukawe (2024), and Mohammed (2010). The Maple 18 software is used to perform all computations on three problems.

4.1. Problem 1

Problem 1 is considered a highly stiff problem.

$$\mu^{(iv)}(x) - x = 0 , \quad h = 0.1: \qquad \text{with Exact Solution: } \mu(x) = \frac{x^5}{120} + x,$$
$$\mu(0) = 0 , \mu'(0) = 1 , \mu''(0) = \mu'''(0) = 0.$$

Table 5. Displays the intended outcome for problem 1

values Associated with x	An Error in the New Approach	Akinnukawe's Error (2024)	Mohammed's Error (2024)
0.1	0.00000	1.38778e-17	7.000024e-10
0.2	0.00000	2.77556e-17	8.9999912e-10
0.3	1.1000 e-20	5.55112e-17	2.5999993e-09
0.4	2.1000 e-20	5.55112e-17	5.100033e-09
0.5	5.1000 e-20	0.00000	7.799979e-09
0.6	1.0100 e-19	1.11022e-16	1.180009e-08
0.7	1.7100 e-19	1.11022e-16	1.180009e-08
0.8	2.8100 e-19	2.22045e-16	1.410006e-08
0.9	4.5100 e-19	2.22045e-16	1.880000e-08
1.0	2.7100 e-19	2.22045e-16	1.008335e-08

4.2. Problem 2

We resolve the ship dynamics physical problem. Consider the fourth-order problem as follows: When a sinusoidal wave of frequency travels along a ship or offshore structure, the resulting fluid motions change over time.

$$\mu^{(iv)} + 3\mu'' + \mu\left(2 + \varepsilon\cos(\Omega t)\right) = 0, \quad t > 0,$$

With the following preconditions in place $\mu(0) = 1, \mu'(0) = 0, \mu''(0) = \mu'''(0) = 0, \ h = \frac{1}{320}$

When $\varepsilon = 0$ for the existence of the theoretical solution: $\mu(t) = 2\cos(t) - \cos\left(t\sqrt{(2)}\right)$

Table 6. Displaying the outcome for problem 2

Values Associated with x	An Error in the New Approach	Victor & Solomon's (2021) Error
0.003125	2.000000 e-20	0.000000
0.00625	5.000000 e-20	1.110223e-16
0.009375	6.000000 e-20	0.000000
0.0125	1.000000 e-20	2.220446e-16
0.015625	2.000000 e-20	7.771561e-16
0.01875	7.000000 e-20	2.886580e-15
0.021875	3.000000 e-20	8.548717e-15
0.0250	4.000000 e-20	2.187139e-14
0.028125	1.000000 e-20	4.951595e-14
0.03125	4.000000 e-20	1.035838e-13

4.3. Problem 3

The fourth-order linear differential equation is examined.

$$\mu^{(iv)} + \mu'' = 0 , \quad h = 0.1 :$$

$$\mu(0) = 0 , \mu'(0) = \frac{-1.1}{72-50\pi} , \mu''(0) = \frac{1}{144-100\pi} , \mu'''(0) = 0\,\frac{1.2}{144-100\pi} , h = \frac{1}{100}$$

With Exact Solution: $\mu(x) = \dfrac{1-x-\cos(x)-1.2\sin(x)}{144-100\pi}$,

Table 7. Displaying the outcome for problem 3

Values Associated with x	An Error in the New Approach	Victor & Solomon's (2021) Error	In Ukpebor et al. (2020) Error
0.01	0.00000	5.4210e-20	8.0000e-20
0.02	0.00000	5.4210e-20	9.1500e-19
0.03	0.000000	2.7105e-19	3.4150e-18
0.04	0.000000	1.0842e-19	8.2220e-18
0.05	0.000000	3.2526e-19	1.5965e-17
0.06	1.00000 e-20	3.2526e-19	2.7404e-17
0.07	1.00000 e-20	3.2526e-19	3.4329e-17
0.08	0.000000	4.3368e-19	3.2551e-17
0.09	1.000000 e-20	2.1684e-19	6.5927e-17
0.10	0.000000	6.5052e-19	1.1919e-16

5. Conclusion

Several fourth derivative numerical problems from Akinnukawe (2024), Mohammed (2010), Victor & Solomon (2021), and Ukpebr et al. (2020) are used to test the created optimized three-step fourth derivative hybrid block approach. The optimal hybrid methods were derived using Scientific Workplace 5.5 software, whereas the implementation was done using Maple 18. The MATLAB 2021a programming language was also used to create the graphical representation, which displays an A-stable zone, as seen above. The data above make it clear that our suggested optimized approaches can handle stiff equations and, in fact, converge more quickly than the current approach. It is evident that the new approach outperforms the current ones when compared to the results obtained in Tables 5, 6, and 7, respectively. If feasible, this approach could be expanded to higher derivatives that provide solutions for both the stiff and oscillatory general ordinary differential equations.

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