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Original Article

The Minimum Surface Area of a Combination of a Right Circular Cylinder, a Spherical Cape, and a Right Circular Cone

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Abstract - At the outset, with a given volume of a right circular cylinder surmounted by a right circular cone on one end and a hemisphere on the other end, the minimum surface area of the combination is determined. The minimum surface area is calculated with the given volume of a right circular cylinder surmounted by a right circular cone on either end of the cylinder. The minimum surface area is calculated with a given volume of a right circular cylinder surmounted by a right circular cone on only one end. The minimum and maximum distances of the points on the major and minor axes of an ellipse from the latter are determined. Finally, the maximum and minimum lengths of a perpendicular drawn from an ellipse to a straight line are determined.

Keywords - Lagrange's multiplier optimization method, Surface area, Right circular cylinder, Spherical cape, Right circular cone.

1. Introduction

Optimization method of Lagrange's multiplier is a technique of finding stationary points of a function subject to one or more equations of constraints followed by evaluating the maximum or minimum value of the function as the situation arises. Many optimization problems with or without solutions are available in almost all Differential [1] Calculus books and the relevant literature [6]. SN Maitra [2-5] innovated several optimization problems and solved them by applying Lagrange's Multiplier. In each section of the present article, the minimum surface area of the compound body is derived from a given volume of a compound body.

In the course of solving the above-mentioned optimization problems, a few of them have showcased the height of the cylinder to be zero, which suggests that with the given volume, the minimum surface area will not contain that of the cylinder. In view of this outcome, an ideal problem is to find the minimum surface of a right circular cylinder surmounted by a right circular cone at one end and a hemisphere at its other end. As far as a significant introduction is concerned, herein are highlighted from Textbook 1 a few problems which have been solved by the present author. A water tank open at the top is of volume V. The task is to find its minimum surface area. Let x and y be the length and breadth of the tank of height z such its volume and surface area according to the question are given by,

$$V=xyz \tag{1}$$

$$S=xy+2z(x+y) \tag{2}$$

Function F involving V, S and Lagrange's Multiplier λ is given by,

$$F=S+\lambda(V-xyz)=xy+2z(x+y)+\lambda(V-xyz) \tag{3}$$

$$\frac{\delta F}{\delta x}=y+2z+\lambda(-yz)=0 \tag{4}$$

$$\frac{\delta F}{\delta y}=x+2z+\lambda(-xz)=0 \tag{5}$$

$$\frac{\delta F}{\delta z}=2(x+y)+\lambda(-xy)=0 \tag{6}$$

Multiplying (4) by x and, (5) by y and then subtracting is obtained.

$$2z(x-y)=0$$

$$\text{Or,} x=y \tag{7}$$

Using (7) in (6),

$$x=y=\frac{4}{\lambda} \tag{8}$$

Eliminating λ between (8) and (4) or (5) is obtained.

$$x=y=\frac{z}{2} \tag{9}$$

Using the optimum values (9) in (1),

$$z^3=4V \qquad \text{Or, } z=(4V)^{\frac{1}{3}} \tag{10}$$

Hence $x=y=\frac{(4V)^{\frac{1}{3}}}{2}$ (11)

Applying the above optimum values of the dimensions in (2), we get the minimum surface area of the tank,

$$S_{min}=xy+2z(x+y)=\frac{(4V)^{\frac{2}{3}}}{4}+2(4V)^{\frac{2}{3}}=\frac{9(4V)^{\frac{2}{3}}}{4}$$

1.1. Example 1

The temperature T on the surface of a sphere $x^2+y^2+z^2=a^2$ is given by equation $T = Kxyz^2$. The aim is to find the maximum temperature on the sphere. In order to apply Lagrange's Multiplier λ is expressed a function F is in terms of T and the constraint equation:

$$F=\log T-\lambda(x^2+y^2+z^2-a^2) \qquad (K=\text{Constant})$$

$$\text{Or, } F=\log x+\log y+2\log z+\log K-\lambda(x^2+y^2+z^2-a^2)$$

$$\frac{\delta F}{\delta x}=\frac{1}{x}-2\lambda x=0$$

$$\frac{\delta F}{\delta y}=\frac{1}{y}-2\lambda y=0$$

$$\frac{\delta F}{\delta z}=\frac{2}{z}-2\lambda z=0$$

The first two of the above three equations give,

$$2\lambda=\frac{1}{x^2}=\frac{1}{y^2}=\frac{2}{z^2}$$

$$\text{Or, } x^2=y^2=\frac{1}{2\lambda} \text{ and } z^2=\frac{1}{\lambda}$$

Substituting, which in the constraint equation is obtained,

$$x^2+y^2+z^2=a^2$$

$$\frac{1}{2\lambda}+\frac{1}{2\lambda}+\frac{1}{\lambda}=a^2$$

$$\lambda=\frac{2}{a^2}, x^2=y^2=\frac{a^2}{4} \text{ and } z^2=\frac{a^2}{2}$$

Hence, in view of the above, the maximum temperature is given by,

$$T_{max}=Kxyz^2=K\frac{a}{2}\frac{a}{2}\frac{a^2}{2}=K\frac{a^4}{8},$$

1.2. Example 2

Find the maximum value of S= $x^2+y^2+\ z^2$ subject to the constraint equation ax+bx+cx=p. In light of earlier examples with Lagrange's multiplier 2λ

$$F==x^2+y^2+\ z^2-2\lambda(ax+bx+cx-p)$$

$$\frac{\delta F}{\delta x}=2x-2\lambda a=0$$

$$\frac{\delta F}{\delta y}=2y-2\lambda b=0$$

$$\frac{\delta F}{\delta z}=2z-2\lambda c=0$$

Which lead to,

$$\lambda=\frac{x}{a}=\frac{y}{b}=\frac{z}{c} \text{ so that,}$$

$$x=a\ \lambda, \qquad y=b\ \lambda, \qquad z=c\ \lambda$$

Which are substituted in the constraint equation to get,

$$a.\ a\ \lambda+b.b\ \lambda+c.\ c\ \lambda=p$$

$$\text{Or, } \lambda=\frac{p}{a^2+b^2+c^2}$$

So that,

$$x=\frac{ap}{a^2+b^2+c^2},\ y=\frac{bp}{a^2+b^2+c^2},\ and\ z=\frac{cp}{a^2+b^2+c^2}$$

Substituting which in the expression of S, the maximum value of the function S *is* is obtained:

$$S_{max} = x^2 + y^2 + z^2 = \frac{p^2}{a^2+b^2+c^2}$$

2. Minimum Surface Area of a Right Circular Cylinder Surmounted by a Hemisphere on its One End and by a Right Circular Cone at the Other End

The task is to determine the minimum Surface area of the aforesaid combination whose volume is given. Thus, if V is the volume of the combination with height H of the cylinder, height h of the cone and common radius r of the cylinder, hemisphere and cone yielding total surface area S, then

$$V = \frac{1}{3}\pi r^2 h + \pi r^2 H + \frac{2}{3}\pi r^3 \tag{12}$$

$$S = \pi r\sqrt{h^2 + r^2} + 2\pi rH + 2\pi r^2 \tag{13}$$

To find the minimum surface area as mentioned above is used Lagrange's Multiplier λ such that function F is given by using (12) and (13):

$$F = \pi r\sqrt{h^2 + r^2} + 2\pi rH + 2\pi r^2 + \lambda\left(V - \frac{1}{3}\pi r^2 h - \pi r^2 H - \frac{2}{3}\pi r^3\right) \tag{14}$$

As per the rule, taking partial derivatives of F with respect to r, H and h respectively and equating to zero, are obtained.

$$\frac{\delta F}{\delta r} = \pi\left(\sqrt{h^2 + r^2} + \frac{r^2}{\sqrt{h^2+r^2}} + 2H + 4r\right) - \lambda\left(\frac{2}{3}\pi rh + 2\pi Hr + 2\pi r^2\right) = 0$$

$$\text{Or, } \left(\sqrt{h^2 + r^2} + \frac{r^2}{\sqrt{h^2+r^2}} + 2H + 4r\right) - \lambda\left(\frac{2}{3}rh + 2Hr + 2r^2\right) = 0 \tag{15}$$

$$\frac{\delta F}{\delta h} = \frac{rh}{\sqrt{h^2+r^2}} - \lambda\left(\frac{1}{3}r^2\right) = 0 \tag{16}$$

$$\frac{\delta F}{\delta H} = 2r - r^2\lambda = 0 \text{ Or, } \lambda = \frac{2}{r} \tag{17}$$

Eliminating λ between (16) and (17),

$$\frac{h}{\sqrt{h^2+r^2}} - \frac{2}{3} = 0 \text{ Or, } 9h^2 = 4(h^2 + r^2) \text{ Or, } h = \frac{2r}{\sqrt{5}} \quad \text{Or} \quad r = \frac{\sqrt{5}}{2}h \tag{18}$$

Employing (17) and (18) in (15),

$$\left(\sqrt{H^2 + r^2} + \frac{r^2}{\sqrt{h^2+r^2}} + 2H + 4r\right) - \lambda\left(\frac{2}{3}rh + 2Hr + 2r^2\right) = \frac{7h}{3} + 2H + 2h\sqrt{5} - \frac{4}{h\sqrt{5}}\left(\frac{2}{3}h\frac{\sqrt{5}}{2}h + 2\frac{\sqrt{5}}{2}hH + 5\frac{h^2}{2}\right) = 0$$

$$\text{Or, } h = 2H,\ r = h\frac{\sqrt{5}}{2} = H\sqrt{5} \tag{19}$$

Applying (19) in (12),

$$V = \frac{1}{3}\pi r^2 h + \pi r^2 H + \frac{2}{3}\pi r^3 = \frac{\pi}{3}10H^3 + 5\pi H^3 + \frac{10}{3}\sqrt{5}\pi H^3 = \frac{(25+10\sqrt{5})\pi H^3}{3}$$

Optimum value of H is given by,

$$\text{Or, } H= \sqrt[3]{\frac{3V}{(25+10\sqrt{5})\pi}} \tag{20}$$

As a consequence of which, (19) gives the other optimum values,

$$h=2\sqrt[3]{\frac{3V}{(25+10\sqrt{5})\pi}} \tag{21}$$

$$r=2\sqrt[3]{\frac{3V}{(25+10\sqrt{5})\pi}} \tag{22}$$

Using (19) and (20) in (13) is obtained the minimum surface area S_{min}:

$$S_{min} = \pi r\sqrt{h^2+r^2} + 2\pi rH + 2\pi r^2 = 4\pi(3+\sqrt{2})H^2 = 4\pi(3+\sqrt{2})\left(\sqrt[3]{\frac{3V}{(25+10\sqrt{5})\pi}}\right)^2 \tag{23}$$

3. Minimum Surface Area of a Right Circular Cylinder Surmounted by a Right Circular Cone on the Other End

In the light of the preceding feature,

$$V=\frac{2}{3}\pi r^2 h+\pi r^2 H \tag{24}$$

$$S=2\pi r\sqrt{h^2+r^2}+2\pi rH \tag{25}$$

Function F containing Lagrange's Multiplier λ is obtained as,

$$F= 2\pi r\sqrt{h^2+r^2}+2\pi rH + \lambda\left(V-\frac{2}{3}\pi r^2 h-\pi r^2 H\right) \tag{26}$$

$$\frac{\delta F}{\delta H} = 0 \rightarrow 2\text{-} r\lambda = 0$$

$$\lambda=\frac{2}{r} \tag{27}$$

$$\frac{\delta F}{\delta r} = 0 \rightarrow \left(\sqrt{h^2+r^2}+\frac{r^2}{\sqrt{h^2+r^2}}+H\right)-\lambda\left(\frac{2}{3}rh+Hr\right)=0 \tag{28}$$

$$\frac{\delta F}{\delta h} = 0 \rightarrow \frac{h}{\sqrt{h^2+r^2}}-\frac{4}{3}=0$$

$$5h^2=4r^2$$

$$\sqrt{5}h = 2r \tag{29}$$

Employing (27) and (29) in (28).

$$\text{Or, }\frac{2r^2+h^2}{\sqrt{h^2+r^2}}+H=\lambda\left(\frac{2}{3}rh+Hr\right)=0$$

$$\text{Or, }\frac{2r^2+h^2}{\sqrt{h^2+r^2}}+H=\frac{4}{3}h+2H$$

$$h=H \tag{30}$$

Combining (29) and (30),

$$r=\frac{\sqrt{5}H}{2} \tag{31}$$

Putting (31) and (32) in (35),

$$V=\frac{2}{3}\pi r^2 h+\pi r^2 H \;=\frac{5}{3}\pi r^2 H \;\;=\frac{5}{3}\pi(\frac{\sqrt{5}H}{2})^2 H \;\; H=\sqrt[3]{\frac{12V}{25}}=h \tag{32}$$

$$\text{Hence} \quad r=\frac{\sqrt{5}}{2}\sqrt[3]{\frac{12V}{25}} \tag{33}$$

Applying the optimum values given by (31) and (33) in (25) obtained the minimum.

3.1. Surface Area

$$S_{min}=2\pi r\sqrt{h^2+r^2}+2\,\pi rH \;\;=2\pi\frac{\sqrt{5}H}{2}\sqrt{H^2+(\frac{\sqrt{5}H}{2})^2}+2\,\pi rH \;\;=2\pi\frac{\sqrt{5}H}{2}\frac{3H}{2}+2\,\pi\frac{\sqrt{5}H}{2}H$$

$$\text{Or,}S_{\min}=5\pi\frac{\sqrt{5}H^2}{2} \;=\frac{5\sqrt{5}}{2}(\sqrt[3]{\frac{12V}{25}})^2\pi \tag{34}$$

4. Minimum Surface Area of a Right Circular Cylinder Surmounted by a Spherical Cape at Both Ends

At the outset, let us determine the volume and surface area of the spherical cap, including height h and radius r of the circular base. If dx is the width of a ring at depth x of the cap cut from a sphere of radius R, then radius y of the ring/disc is given by geometry:

$$y^2=R^2-(R-x)^2=(2R\text{-}x)x \tag{35}$$

By use of (35), the volume of the cap is obtained as,

$$V_1=\pi\int_0^h(2R-x)x\,dx=\pi(Rh^2-\frac{h^3}{3}) \tag{36}$$

In order to find the surface area of the ring, let the line joining the centre of the sphere to line element $Rd\theta$ makes angle θ with the diameter. Then, the radius of the ring is $R\sin\theta$. To cover the Surface area of the cape θ varies from 0 to α such that,

$$\cos\alpha=\frac{R-h}{R} \tag{37}$$

Hence, the surface area is given by considering below and above the diameter.

$$S=\int_0^{\alpha}2\,\pi R\sin\theta Rd\theta=2\pi R^2(1-\cos\alpha)=2\pi Rh \tag{38}$$

Further, by geometry, the radius r of the circular base of the cape is given by,

$$r^2=R^2\text{-}(R-h)^2\text{=}2Rh\text{-}h^2$$

$$\text{Or } R=\frac{h^2+r^2}{2h} \tag{39}$$

In consequence of (39), (36) is amended as,

$$V_1=\pi\left(\frac{r^2h}{2}+\frac{h^3}{6}\right) \tag{40}$$

Hence, the volume of the right circular cylinder surmounted by the two spherical capes is,

$$V=\pi r^2H+2\pi\left(\frac{r^2h}{2}+\frac{h^3}{6}\right) \tag{41}$$

Combining (39) and (40) gives the surface area of the two capes.

$$S=2\pi(h^2+r^2) \tag{42}$$

Using (40) and (41) to find the minimum surface area as depicted above, we construct a function F involving Lagrange's Multiplier λ:

$$F=2\pi(h^2+r^2)+2\pi rH+\lambda\left\{V-2\pi\left(\frac{r^2h}{2}+\frac{h^3}{6}\right)-\pi r^2H\right\} \tag{43}$$

$$\frac{\delta F}{\delta H}=0 \text{ gives }=\lambda=\frac{2}{r} \tag{44}$$

Using (44)

$$\frac{\delta F}{\delta h}=0 \text{ gives } 2h-\frac{2}{r}\left(\frac{r^2}{2}+\frac{h^2}{2}\right)=0$$

$$\text{Or, } h=r \tag{45}$$

$$\frac{\delta F}{\delta r}=2r+H-\lambda(rh+rH)=0$$

$$0r,\ 2r+H-\frac{2}{r}(rh+rH)=0$$

$$\text{Or, } H=0 \tag{46}$$

Which suggests that any solid of a given volume is to be moulded into a sphere to produce thereof minimum surface area.

5. Minimum Surface Area of a Right Circular Cylinder Surmounted by a Right Circular Cone

If H and h are the heights of the cylinder and the cone and r, their common radius, volume and surface area are given by,

$$V=\frac{2}{3}\pi r^2h+\pi r^2H \tag{47}$$

$$S=2\pi r\sqrt{h^2+r^2}+2\pi rH+\pi r^2 \tag{48}$$

A function F involving V, S and Lagrange's Multiplier λ is written as,

$$F=2\pi r\sqrt{h^2+r^2}+2\,\pi rH+\pi r^2+\lambda(V-\frac{2}{3}\pi r^2h-\pi r^2H) \tag{49}$$

$$\frac{\delta F}{\delta H}=2r-\lambda r^2=0 \text{ gives } \lambda=\frac{2}{r} \tag{50}$$

$$\frac{\delta F}{\delta r}=\frac{2r^2+h^2}{\sqrt{h^2+r^2}}+2H+2r-\lambda\left(\frac{2}{3}rh+2Hr\right)=0$$

Which by use of (50) gives,

$$\frac{2r^2+h^2}{\sqrt{h^2+r^2}}-2H-\frac{4h}{3}=o$$

$$2r^2+h^2=\left(2H+\frac{4h}{3}\right)\sqrt{h^2+r^2} \tag{51}$$

$$\frac{\delta F}{\delta h}=\frac{hr}{\sqrt{h^2+r^2}}-\frac{2}{r}\frac{r^2}{3}=0$$

Simplifying and squaring both sides,

$$4r^2=5h^2$$

$$r=\frac{\sqrt{5}}{2}h \tag{52}$$

6. Minimum and Maximum Distances of an Ellipse from a Point on the Major Axis

Let (h, 0) be the above-mentioned point lying on the major axis, vis-à-vis X-axis of a right-handed system XOY, of an ellipse.

$$\frac{x}{a^2}^2+\frac{y}{b^2}^2=1 \tag{53}$$

Which can be written in parametric form,

$$x=a\cos\theta\,,\quad y=b\sin\theta \tag{54}$$

Which represent coordinates of any point on the ellipse (56), θ being the parameter. With this point the distance D between point (h, 0) and the ellipse (56) is given by,

$$D^2=(a\cos\theta-h)^2+(b\sin\theta\,)^2=(a\cos\theta)^2+(b\sin\theta\,)^2-2ah\cos\theta+h^2 \tag{55}$$

For maximum and minimum distance, D.

$$\frac{dD^2}{d\theta}=-2a(a\cos\theta-h)\sin\theta+2b^2\sin\theta\cos\theta=0 \tag{56}$$

$$\text{Or, } a(a\cos\theta-h)=b^2\cos\theta \tag{57}$$

$$\text{and } \sin\theta=\ \ 0$$

$$\text{Or, } \cos\theta=\frac{ah}{a^2-b^2},\ \ \theta=0\,,\pi \tag{58}$$

Which gives one optimum value of θ.

Again from (59),

$$\frac{d^2D^2}{d^2\theta^2} = 2a^2\sin^2\theta - 2a(a\cos\theta - h)\cos\theta + 2b^2\cos 2\theta$$

$$= -2(a^2 - b^2)\cos 2\theta + 2ah\cos\theta = -2(a^2 - b^2) - 2ah < 0 \text{ when } \theta = \pi \tag{59}$$

This suggests that the distance D is maximum when $\theta = \pi$ and, as such, the maximum.

Distance by virtue of (58) is obtained as,

$$D_{max} = a + h \tag{60}$$

This can also be observed by drawing the figure of the ellipse.

From (57), another optimum value, $\cos\theta = \frac{ah}{a^2-b^2}$, when substituted in (59) leads to,

$$\frac{d^2D^2}{d^2\theta^2} = 2a^2\sin^2\theta - 2a(a\cos\theta - h)\cos\theta + 2b^2\cos 2\theta = -2(a^2 - b^2)\cos 2\theta + 2ah\cos\theta$$

$$= 2(a^2 - b^2)(1 - 2\cos^2\theta) + 2ah\cos\theta = 2(a^2 - b^2)\left(1 - 2\left(\frac{ah}{a^2-b^2}\right)^2\right) + 2ah\frac{ah}{a^2-b^2}$$

$$= 2(a^2 - b^2) - 4\frac{(ah)^2}{a^2-b^2} + 2\frac{(ah)^2}{a^2-b^2} \quad a>h,\ a^2 > h^2 < a^2 \ -h^2 > -a^2$$

$$= 2\{(a^2 - b^2) - \frac{(ah)^2}{a^2-b^2}\} = 2\left((ae)^2 - \frac{a^2h^2}{(ae)^2}\right) > 0 \tag{61}$$

Because, $\cos\theta = \frac{ah}{a^2-b^2} = \frac{ah}{(ae)^2} < 1$, Where e is the eccentricity of the ellipse.

$$\text{Or}, -\frac{a^2h^2}{(ae)^4} > -1$$

$$-\frac{a^2h^2}{(ae)^2} > -(ae)^2 \tag{62}$$

Hence, the optimum value of θ for minimum distance D_{min} is given by,

$$\cos\theta(opt) = \frac{ah}{a^2-b^2} = \frac{ah}{a^2e^2} = \frac{h}{ae^2} < 1 \quad \text{and } e<1 \tag{63}$$

Substituting (66) in (58),

$$(D^2)min = \left(\frac{h}{e^2} - h\right)^2 + b^2\left(1 - \frac{h^2}{a^2e^4}\right) = \frac{h^2}{e^4} + h^2 - \frac{2h^2}{e^2} + a^2(1-e^2)\left(1 - \frac{h^2}{a^2e^4}\right)$$

$$= \frac{h^2}{e^4} + h^2 - \frac{2h^2}{e^2} + a^2(1-e^2) - (1-e^2)\frac{h^2}{e^4} \tag{64}$$

$$= h^2 - \frac{h^2}{e^2} + a^2(1-e^2) = (1-e^2)\left(a^2 - \frac{h^2}{e^2}\right) \tag{65}$$

Because (63) h cannot be chosen equal to the focal length, i.e., (ae)

7. Maximum and Minimum Distances of a Point Lying on the Minor Axis of the Ellipse from the Ellipse

Let (0, k) be the point on the minor axis of the same ellipse (53); the distance D between the former point and any point $(a\cos\theta,\ b\sin\theta)$ on the ellipse is given by

$$D^2 = (a\cos\theta)^2 + (b\sin\theta - k)^2 \tag{66}$$

For maximum and minimum of the distance D,

$$\frac{dD^2}{d\theta} = -2\{(a^2 - b^2)\sin\theta + bk\}\cos\theta = 0 \tag{67}$$

$$\cos\theta = o \tag{68}$$

$$\theta = \frac{\pi}{2}$$

Another value of θ is given from (67) as

$$(a^2 - b^2)\sin\theta + bk = 0$$

$$\sin\theta = -\frac{bk}{a^2-b^2} \quad = -\frac{bk}{a^2e^2} \tag{69}$$

From (68),

$$\frac{d^2D^2}{d\theta^2}=-2(ae)^2\cos2\theta+2bk\sin\theta \tag{70}$$

$$\frac{d^2D^2}{d\theta^2}>0$$

$$\text{when } \theta = \frac{\pi}{2} \tag{71}$$

Vide (70),

Which leads to minimum distance obtained from (68):

$$D_{min}= b - k \tag{72}$$

Which can also be ascertained by drawing the relevant figure.

Using (69) in (70) is obtained

$$\frac{d^2D^2}{d\theta^2}=-2(ae)^2\cos2\theta+2bk\sin\theta \quad =-2(ae)^2(1 - 2\frac{(bk)^2}{a^4e^4})-2\frac{(bk)^2}{a^2e^2}$$

$$=-2(ae)^2(1 - \frac{(bk)^2}{a^4e^4}) = -2(ae)^2(1 - (\sin\theta)^2) < 0 \tag{73}$$

Which suggests that distance D is maximum due to (70) and on account of (68) and (70), the maximum distanceD_{max} is given by

$$D_{max}^2=(a\cos\theta)^2 + (b\sin\theta - k)^2=a^2(1 - \sin^2\theta) + (b\sin\theta - k)^2$$

$$= a^2(1 - \frac{(bk)^2}{(ae)^4}) + \left(-\frac{b^2k}{(ae)^2} - k\right)^2 = a^2(1 - \frac{(bk)^2}{(ae)^4}) + \frac{b^4k^2}{a^4e^4} + 2\frac{b^2k^2}{(ae)^2} + k^2$$

$$= a^2 - \frac{b^2k^2}{a^4e^4}(a^2 - b^2) + 2\frac{b^2k^2}{(ae)^2} + k^2 \quad = a^2 - \frac{(bk)^2}{a^4e^4}(ae)^2 + 2\frac{b^2k^2}{(ae)^2} + k^2$$

$$= a^2 + \frac{b^2k^2}{(ae)^2} + k^2 \quad = a^2 + \frac{a^2(1-e^2)k^2}{(ae)^2} + k^2 \tag{74}$$

$$D^2_{max} = a^2 + \frac{k^2}{e^2} \tag{75}$$

8. Minimum and Maximum Lengths of the Perpendiculars from an Ellipse to a Straight Line

Length of the perpendicular drawn from a point $(a\cos\theta, b\sin\theta)$lying on an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{76}$$

To a straight line with positive constants.

$$Ax+By+C=0 \tag{77}$$

Can be written as,

$$p = \frac{Aa\cos\theta + Bb\sin\theta + C}{\sqrt{A^2+B^2}} \tag{78}$$

For maximum and minimum of p,

$$\frac{dp}{d\theta} = \frac{-Aa\sin\theta + Bb\cos\theta}{\sqrt{A^2+B^2}} = 0 \tag{79}$$

Which gives,

$$\tan\theta = \frac{Bb}{Aa} \tag{80}$$

As such, the optimum values are,

$$\sin\theta = \pm\frac{Bb}{\sqrt{(Aa)^2+(Bb)^2}} \text{ and } \cos\theta = \pm\frac{Aa}{\sqrt{(Aa)^2+(Bb)^2}} \tag{81}$$

$\frac{d^2p}{d\theta^2} = \frac{-Aa\cos\theta - Bb\sin\theta}{\sqrt{A^2+B^2}} < 0$ for positive values of $\sin\theta$ and $\cos\theta$ from (81).

Then p is maximum, and using (81) in (79) is obtained.

$$p_{max} = = \frac{Aa\cos\theta + Bb\sin\theta + C}{\sqrt{A^2+B^2}} = \frac{\sqrt{A^2a^2+B^2b^2}+C}{\sqrt{A^2+B^2}}$$

$\frac{d^2p}{d\theta^2} = \frac{Aa\cos\theta + Bb\sin\theta}{\sqrt{A^2+B^2}} > 0$ for negative values of $\sin\theta$ and $\cos\theta$ from (81).

Hence, the minimum value of p is,

$$p_{min} = \frac{C-\sqrt{A^2a^2+B^2b^2}}{\sqrt{A^2+B^2}} \quad (82)$$

9. Discussion and Conclusion

To optimize, i.e., to find the maximum and/or minimum value of a function of several variables subject to one or more constraints, use Lagrange's Multiplier. With no constraint(s), the maximum and/or minimum value of a function of several, preferably one or two variables, is derived in almost all textbooks of Differential Calculus.

For example, it is used to find the maximum and/or minimum value of a function.

$$F(x,y)=x^3 + y^3 - 3axy \quad (83)$$

By the usual method given in almost all textbooks of Calculus,

$$\frac{\delta F}{\delta x}=3x^2-3ay=0 \quad (84)$$

$$\frac{\delta F}{\delta y}=3y^2-3ax=0 \quad (85)$$

$$\frac{\delta^2 F}{\delta x^2} = 6x \quad (86)$$

$$\frac{\delta^2 F}{\delta y^2} = 6y \quad (87)$$

$$\frac{\delta^2 F}{\delta x \delta y} = \frac{\delta^2 F}{\delta x \delta y} = 3a > 0 \quad (88)$$

Solving (83) and (84) is obtained.

$$x=y=a \quad (89)$$

$$\frac{\delta^2 F}{\delta x^2} = 6a = \frac{\delta^2 F}{\delta y^2} \qquad \text{at } (a, a) \quad (90)$$

Which, together with (87), implies that the function (82) attains the minimum $-a^3$ at (a, a), i.e. when the values of x, y are given by (88).

Now, let us tackle a real-world problem: A cylindrical tank is filled with water, and water is supplied to a number of tanks on the terraces of buildings in a certain area of a city. The task is to find the minimum surface area of each tank fully filled with water, leading to the minimum surface area of the metallic sheet on which the terrace tanks are constructed.

Let V be the volume of water filling all n number of tanks while x_r, y_r, z_r *are the dimensions of* r_{th} *tank* .Then,

$$V=\sum_1^n x_r\, y_r\, z_r \quad (91)$$

If S is the surface area of the metallic sheet required to construct the tanks,

$$S=2\sum_1^n (x_r\, y_r + y_r\, z_r + z_r x_r) \quad (92)$$

With given volume V, in order to find the minimum surface area, is written a function F is applied.

Lagrange's Multiplierλ and using (90) and (91) is obtained,

$$F=S+\lambda(V - \sum_1^n x_r\, y_r\, z_r) = 2\sum_1^n (x_r\, y_r + y_r\, z_r + z_r x_r + \lambda(V - \sum_1^n x_r\, y_r\, z_r) \quad (93)$$

For a minimum of F, i.e. of S, (r=1, 2, 3,…n),

$$\frac{\delta F}{\delta x_r} = \sum_1^n (y_r + z_r) - \lambda \sum_1^n y_r\, z_r) = 0 \quad (94)$$

$$\frac{\delta F}{\delta y_r} = \sum_1^n (x_r + z_r) - \lambda \sum_1^n x_r\, z_r) = 0 \quad (95)$$

$$\frac{\delta F}{\delta z_r} = \sum_1^n (y_r + x_r) - \lambda \sum_1^n y_r\, x_r) = 0 \quad (96)$$

Solving the above equations: subtracting, (93)-(94), (94)-(95)

$$y_r = z_r = x_r = a \text{ (say) } (r=1, 2, 3.....n) \quad (97)$$

Which means that with the supply of water from the mother (bigger) water tank into terrace tanks, each terrace tank has to be a cubic of the same dimensions to ensure a minimum surface area of the metallic sheet comprising all the cubical water tanks rather than constructing the terrace water tanks of different dimensions. Obviously, this venture reduces the cost of the metallic sheet used to construct the water tanks without reducing the water supply. Given volume V of supply water, each side of length each of n cubical tanks depicted above) gives a relationship.

$$na^3 = V \quad (98)$$

$$\text{Or, } a=\sqrt[3]{\frac{V}{n}} \quad (99)$$

The minimum surface $area\ S_{min}$ of the metallic sheet=minimum surface area of all the terrace tanks:

$$S_{min} = 6na^2 \quad (100)$$

Eliminating a between (99) and (100),

$$S_{min} = 6n(\sqrt[3]{\frac{V}{n}})^2 \quad (101)$$

Besides Lagrange's Multiplier, a branch of Mathematics called Calculus of variation is involved in Physics, Engineering and some other fields to determine the maximum or minimum value of a function in the form of a definite integral:

$$I=\int_a^b f(y, \frac{dy}{dt}, x)dx \quad (102)$$

Which needs to satisfy the Euler-Lagrange differential equation.

$$\frac{\delta f}{dy} - \frac{d}{dt}(\frac{\delta f}{\delta \dot{y}}) = 0 \quad \text{where} \quad \frac{dy}{dt} = \dot{y} \quad (103)$$

Further, Brachistochrome is a smooth curve connecting two points, A and B, yielding a path along which a particle slides in the shortest time under the gravitational force. This is derived by using the Calculus of Variation, a technique of maximization/ minimization.

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