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Original Article

Projectile Motion from a Moving Platform to a Stationary Point on the Same Level Ground

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Abstract - A bullet is fired from a platform moving forward to study its projectile motion, and thereafter, it moves backwards with the same uniform velocity and strikes a stationary object on level ground. In either case, the maximum horizontal range is determined. The case is also studied when the projection velocity equals the velocity of movement of the platform. Two problems on projectile motion, respectively, with a prescribed time of flight and with a prescribed horizontal range involving grazing the top of a tower, are innovated and solved. For the third problem, two projectiles are launched at a certain interval of time, triggering a collision between them. The interval of time observed and the location of the interception are determined. As regards the fourth problem, a shot is fired from a stationary gun position on the bank of a river and hits a gunboat sailing with a uniform velocity parallel to the opposite bank, while the task is to find the minimum velocity of firing the shot.

Keywords - Velocity, Projectile motion, Fired bullet, Horizontal range, Trajectory.

1. Introduction

Textbook contains many problems with and without thereof solutions [1]. MS Ganagi and VP Singh [2] analysed projectile motion, including atmospheric drag varying as the square of the velocity, but considered small variations of the elevation angle and ultimately obtained an analytical solution to the projectile trajectory. He obtained a closed-form solution for trajectories of ballistic projectiles, neglecting variation of the elevation angle in small intervals of time along the trajectory [2]. His closed-form solution has been used to develop an algorithm for a lead angle computation and underscored numerical experiment and error analysis. SN Maitra present author, solved several projectile problems that arise in a cricket game on account of bowled-out, caught-out, striking boundary and over-boundary [3-6].

SN Maitra, the present author, also solved projectile problems posed due to the throwing of a tiny tool by an astronaut towards another astronaut in a space vehicle falling freely or in an orbiting spacecraft. He published a paper related to the re-entry of a ballistic missile into the atmosphere where the drag obeys the square law of velocity and the density of the air decreases exponentially with the increase in height. In this feature, a bullet is fired from a platform firstly moving forward and secondly moving backwards and strikes, in either case, a stationary object on the level ground. In both cases, the maximum horizontal range is derived. Finally, four problems on projectile motion are innovated and solved by the present author. As far as the Introduction is concerned, a textbook problem given as an exercise is solved herein by the present author.

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2. Statement of the Problem

From a gun, a cannon is fired from a moving platform, and the ranges of the shot are observed to be R and S when the platform is moving forward and backward, respectively, with velocity v. Prove that the elevation of the gun is $tan^{-1}\{\frac{g}{4v^2}.\frac{(R-S)^2}{R+S}\}$.

3. Solution to the Problem

If u is the velocity of projection of the cannon at an angle α to the horizontal relative to the platform moving forward with velocity v, the horizontal and vertical components of the actual velocity of the bullet are (ucos α+v) and usin α, which lead to the time T of flight and horizontal range R:

$$T=\frac{2u\ sin\alpha}{g} \tag{1}$$

$$R=\frac{(u\ cos\alpha+v)2u}{g}\ sin\alpha\ =\frac{(u^2\ sin2\alpha+2uvsin\alpha)}{g} \tag{2}$$

$$S=\frac{(u^2\ sin2\alpha-2uvsin\alpha)}{g} \tag{3}$$

Hence, R-S = (4uv$sin\alpha$)/g, which leads to,

$$\frac{(R-S)^2}{R+S}=\frac{\{\frac{(4uvsin\alpha)}{g}\}^2}{\frac{(2u^2\ sin2\alpha)}{g}}=\frac{4v^2}{g}tan\alpha \tag{4}$$

$$\text{Or,}\ \ tan\alpha\ \ =\frac{(R-S)^2}{R+S}.\frac{g}{4v^2} \tag{5}$$

Maximum horizontal ranges corresponding to moving forward and backward. In this paper are obtained the maximum ranges R_{max} and S_{max} in both of the above cases. For maximum or minimum of R and S,

$$\frac{dR}{d\alpha}=\frac{2u}{g}(ucos2\alpha+vcos\alpha)=0 \tag{6}$$

$$\frac{dS}{d\alpha}=\frac{2u}{g}(ucos2\alpha-vcos\alpha)=0 \tag{7}$$

Which turn out to be,

$$2ucos^2\alpha+vcos\alpha-u=0 \tag{8}$$

$$2ucos^2\alpha-vcos\alpha-u=0$$

For the forward motion of the platform the optimum angle of projection is given by solving (8).

$$ucos2\alpha\ =-vcos\alpha \tag{9}$$

$$cos\alpha=\frac{-v+\sqrt{v^2+8u^2}}{4u} \tag{10}$$

Using (10) in (9),

$$cos2\alpha = \frac{-v(}{4}\frac{-v+\sqrt{v^2+8u^2})}{u^2} \tag{11}$$

$$\sin\alpha = \sqrt{(1+cos\alpha)(1-cos\alpha)} \quad = \sqrt{\frac{(4u-v+\sqrt{v^2+8u^2})}{16u^2}\,(4u+v-\sqrt{v^2+8u^2})} \tag{12}$$

$$= \frac{1}{4u}\sqrt{16u^2-2v^2-8u^2)+2v\sqrt{v^2+8u^2}}$$

Employing (10) and (11) in (2) is obtained,

$$R_{max} = \frac{(u\,cos\alpha+v)2u}{g}\,sin\alpha = \frac{(3v+\sqrt{v^2+8u^2})}{8g}\cdot\sqrt{16u^2-2v^2-8u^2)+2v\sqrt{v^2+8u^2}} \tag{13}$$

Obviously, replacing v with -v is obtained,

$$S_{max} = \frac{(-3v+\sqrt{v^2+8u^2})}{8g}\cdot\sqrt{16u^2-2v^2-8u^2)-2v\sqrt{v^2+8u^2}} \tag{14}$$

If u=v, i.e. the velocity of the moving platform in the forward direction is the same as that of projection, (10) yields,

$$\text{Cos}\ \alpha = \frac{1}{2} \qquad \text{Or, } \alpha = 60^0 \tag{15}$$

If u=3v, from (10), $cos\alpha = \frac{-v+\sqrt{v^2+8u^2}}{4u} = \frac{1}{3}$ Or $\alpha = 19^0$ approximately by use of a log table. (16)

But in case of moving in the backward direction, v is to be replaced by -v, (10) with u=v, gives cos α =1 Or, α=0 leading to no horizontal movement of the bullet, i.e. S=0.

In case of moving backwards with u=3v, v is replaced by -v so that (10) gives.

$$cos\alpha = \frac{1}{2}\ \text{ie}\ \alpha = 60^0 \tag{17}$$

3.1. Problem No. 1 on Projectile Motion and Its Solution

A stone is thrown to graze the top of a tower at a distance x on the same level ground, obtaining a time T of flight. Find the maximum height of the tower, g being the acceleration due to gravity.

Let the stone be thrown with a velocity u at an angle. α to the horizontal from the level ground and graze the tower of height h in time t. Then

$$h = ut\sin\alpha - \frac{1}{2}gt^2 \tag{18}$$

$$x = ut\cos\alpha \tag{19}$$

$$T = \frac{2u\sin\alpha}{g} \tag{20}$$

Eliminating t between (18) and (19),

$$h=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha} \qquad (21)$$

Eliminating u between (20) and (21),

$$h=x\tan\alpha-\frac{gx^2}{2\cos^2\alpha}\left(\frac{2\sin\alpha}{gT}\right)^2=x\tan\alpha-\frac{2x^2}{gT^2}\tan^2\alpha \qquad (22)$$

For the maximum height of the grazing point with respect to the angle of projection α,

$$\frac{dh}{d(\tan\alpha)}=x-\frac{4x^2}{gT^2}\tan\alpha=0$$

$$\text{Or, }\tan\alpha=\frac{gT^2}{4x} \qquad (23)$$

Combining which with (20), we get,

$$u_{0pt}=\frac{gT}{2\sin\alpha}=\frac{gT\sqrt{1+tan^2\alpha}}{2\tan\alpha}=\frac{gT\sqrt{\frac{1}{tan^2\alpha}+1}}{2}=\frac{gT\sqrt{\frac{16x^2}{g^2T^4}+1}}{2} \qquad (24)$$

Substituting (23) in (22) obtained the maximum height of the tower which can be cleared by the stone is projected at an angle given by (23):

$$h_{max}=x\,\frac{gT^2}{4x}-\frac{2x^2}{gT^2}\left(\frac{gT^2}{4x}\right)^2=\frac{gT^2}{8} \qquad (25)$$

Now we find the maximum height of the tower with respect to the time of flight: From (22) for the maximum value of h,

$$\frac{dh}{dx}=\tan\alpha-\frac{4x}{gT^2}tan^2\alpha=0$$

$$x=\frac{gT^2}{4\tan\alpha} \qquad (26)$$

Putting which in (22) is obtained the maximum height[1] w.r.t. to the time T of flight:

$$\text{height}^1=x\tan\alpha-\frac{2x^2}{gT^2}tan^2\alpha=\frac{gT^2\tan\alpha}{4\tan\alpha}-\frac{2\left(\frac{gT^2}{4\tan\alpha}\right)^2}{gT^2}tan^2\alpha=\frac{gT^2}{8} \qquad (27)$$

3.2. Problem No. 2

In this design, the stone executes a horizontal range R with the other parameters remaining the same as in the previous Problem No. 1. With a view to (22), we get

$$h=x\tan\alpha-\frac{gx^2}{2u^2cos^2\alpha} \qquad (28)$$

$$R=\frac{u^2}{g}\sin2\alpha \qquad (29)$$

Eliminating u between (28) and (29) is obtained.

$$h = x\tan\alpha - \frac{x^2}{2\cos^2\alpha}.\frac{1}{R}\sin 2\alpha \ = x(1-\frac{x}{R})\tan\alpha \tag{30}$$

For maximum height h_2 of the grazing point with respect to x is,

$$x_{opt} = \frac{R}{2} \ ; \ h_2 = \frac{R}{4}\tan\alpha \tag{31}$$

3.3. Problem No. 3 and Its Solution

A stone is thrown with a velocity u_1 at an angle α_1 to the horizontal. Another stone is thrown with a velocity u_2 at an angle α_2 to the horizontal from the same level ground after time t_1 of projection of the first.

If the two stones are observed to collide with each other, find the time that elapses at the instant of collision, the location of the collision point and the distance of the second point of projection from the vertical plane of projection of the first one.

If t is the time of flight of the second stone for the collision with the first at height h and at horizontal distance x on the level ground and a is the distance of the point of projection of the second stone from the plane of projection of the first, considering the vertical motion the equations of motion are,

$$h = u_1(t+t_1)\sin\alpha_1 - \frac{1}{2}g(t+t_1)^2 \tag{32}$$

$$= u_2 t\sin\alpha_2 - \frac{1}{2}gt^2 \tag{33}$$

Which lead to,

$$(u_1\sin\alpha_1 - u_2\sin\alpha_2 - gt_1)t = \frac{1}{2}gt_1^2 \quad \text{Or,} \quad t = \frac{\frac{1}{2}gt_1^2}{u_1\sin\alpha_1 - u_2\sin\alpha_2 - gt_1} \tag{34}$$

Considering the horizontal motion,

$$x = u_1(t+t_1)\cos\alpha_1 \tag{35}$$

$$\sqrt{x^{2+}a^2} = u_2 t\cos\alpha_2 \tag{36}$$

Squaring (35) and (36) and subtracting is obtained.

$$a^2 = (u_2 t\cos\alpha_2)^2 - \{u_1(t+t_1)\cos\alpha_1\}^2 \tag{37}$$

Eliminating t with (34) from (37), (35) and (32) are obtained.

$$a^2 = (u_2\frac{\frac{1}{2}gt_1^2}{u_1\sin\alpha_1 - u_2\sin\alpha_2 - gt_1}\cos\alpha_2)^2 - \{u_1(\frac{\frac{1}{2}gt_1^2}{u_1\sin\alpha_1 - u_2\sin\alpha_2 - gt_1} + t_1)\cos\alpha_1\}^2 \tag{38}$$

$$x = u_1(\frac{\frac{1}{2}gt_1^2}{u_1\sin\alpha_1 - u_2\sin\alpha_2 - gt_1} + t_1)\cos\alpha_1 \tag{39}$$

$$h = u_2(\frac{\frac{1}{2}gt_1^2}{u_1\sin\alpha_1 - u_2\sin\alpha_2 - gt_1})\sin\alpha_2 - \frac{1}{2}g(\frac{\frac{1}{2}gt_1^2}{u_1\sin\alpha_1 - u_2\sin\alpha_2 - gt_1})^2 \tag{40}$$

3.4. Problem No. 4 and Its Solution

A gunboat is sailing with a velocity v parallel to a bank in a river. When the line of sight of the gunboat from a gun position on the opposite bank is at right angles to the former bank, a shot is fired with a velocity u at an elevation α and hits the gunboat. If the breadth of the river is a, find the minimum velocity of firing the shot to strike the target. If T is the time of flight of the shot, by geometry,

$$a^2 + (vT)^2 = (ucos\alpha T)^2 \tag{41}$$

$$T=\frac{2usin\alpha}{g} \tag{42}$$

Eliminating T between (41) and (42), we get,

$$a^2 + \{v^2 - (ucos\alpha)\}^2(\frac{2usin\alpha}{g})^2=0$$

$$Or, u^4\ (cos\alpha)^2\text{-}v^2u^2 - \frac{a^2g^2}{4sin^2\alpha}=0 \tag{43}$$

$$Or\ \ u^2=\frac{v^2+\sqrt{v^4+a^2g^2cot^2\alpha}}{2cos^2\alpha} \tag{44}$$

Eliminating α from (41) and (42),

$$a^2 + (v^2 - u^2cos^2\alpha)T^2 = 0\ Or,\ a^2 + \{v^2 - u^2(1 - \frac{g^2T^2}{4u^2})\}T^2 = 0$$

$$Or,\ u^2 - v^2 - \frac{g^2T^2}{4} = \frac{a^2}{T^2} = 0,\ \ u^2 = \frac{a^2}{T^2} + \frac{g^2T^2}{4} + v^2 \tag{45}$$

For minimum velocity of projection u with respect to T,

$$\frac{du^2}{d(T^2)} = \frac{-a^2}{T^4} + \frac{g^2}{4} = 0$$

$$T=\sqrt{2\frac{a}{g}} \tag{46}$$

Which gives the optimum time of flight.

Employing (42) in (41) is obtained the minimum velocity u_{min} of projection:

$$u_{min}^2 = \frac{ag}{2} + \frac{ag}{2} + v^2 = ag+v^2 \tag{47}$$

Using (43) and (42) in (38), we get the optimum angle α_{opt} of projection:

$$sin\ \alpha_{opt} = \frac{gT_{opt}}{2u_{min}} = \frac{1}{2}\sqrt{\frac{2ag}{ag+v^2}} \tag{48}$$

3.5. Problem No. 5 and Its Solution

A stone is thrown from a level ground to graze two pillars of the same height, h, acquiring the total time of flight T and horizontal range R. Find the distance the pillars are apart.

If the stone is projected with a velocity u at an angle $\alpha$to the horizontal taking time t to rise the height h, h=usin $\alpha\, t - \frac{1}{2} g t^2$. (49)

$$Or, t^2 \text{-} \frac{2usin\ \alpha}{g} t + \frac{2h}{g} = 0 \quad (50)$$

If t_1 and t_2 are the roots of the Equation (50), the time T of flight and the time T' taken to pass the pillars by grazing are given by,

$$T = t_1 + t_2 = \frac{2usin\ \alpha}{g} \text{ and } t_1 t_2 = \frac{2h}{g}$$

$$T' = t_1 - t_2 = \sqrt{(t_1 + t_2)^2 - 4t_1 t_2} = \sqrt{(\frac{2usin\ \alpha}{g})^2 - \frac{8h}{g}}$$

$$= \frac{2}{g}\sqrt{(u^2 sin^2\ \alpha - 2gh} \quad (51)$$

(51) is also obtained in another way:

The vertical velocity of the stone after attaining the first pillar in front of the point of projection is,

$$v' = \sqrt{(u^2 sin^2\ \alpha - 2gh} \quad (52)$$

Which is the initial velocity for the horizontal range a. Hence, the distance between the two pillars is obtained as,

$$a = (ucos\ \alpha)T' = \frac{2ucos\ \alpha}{g}\sqrt{u^2 \sin^2\ \alpha - 2gh} \quad (53)$$

In order to find u in terms of R and T, we write,

$$R = \frac{u^2 sin2\ \alpha}{g} \quad \text{and } T = \frac{2usin\ \alpha}{g} \quad (54)$$

$$\text{Or, } usin\ \alpha = \frac{gT}{2} \quad (55)$$

$$\frac{gT^2}{2R} = \tan\ \alpha \quad (56)$$

$$Or,\ u = \frac{gT}{2sin\ \alpha} = \frac{gT}{2}\sqrt{1 + cot^2\alpha} = \frac{gT}{2}\sqrt{1 + \frac{4R}{g^2T^4}^2}$$

For the maximum distance given by (53),

$$\frac{da}{d\alpha} = \frac{d}{d\alpha}\left(\frac{2ucos\ \alpha}{g}\sqrt{u^2 sin^2\ \alpha - 2gh}\right) = \frac{2u}{g}\left\{-sin\ \alpha\sqrt{u^2 sin^2\ \alpha - 2gh} + \cos\ \alpha \frac{u^2 sin\ 2\alpha}{2\sqrt{u^2 sin^2\ \alpha - 2gh}}\right\} = 0$$

$$sin\ \alpha(u^2 sin^2\ \alpha - 2gh) + u^2 cos^2\ \alpha sin\ \alpha = 0$$

Discarding sin α = 0, which hypothetically suggests the minimum distance = 0.

Or, $\cos 2\alpha = \frac{2gh}{u^2}$ (57)

Or, $1\text{-}2sin^2\ \alpha = 2cos\ ^2\alpha - 1 = \frac{2gh}{u^2}$

Or, $sin^2\ \alpha = \frac{1}{2}(1 - \frac{2gh}{u^2})\ \ and\ \ cos^2\ \alpha = \frac{1}{2}(1 + \frac{2gh}{u^2})$ (58)

Which gives the optimum angle of projection.

Using (58) in (53) is obtained the maximum distance. a_{max} Between the two pillars:

$$a_{max} = \frac{2u\cos\alpha}{g}\sqrt{u^2 sin^2\ \alpha - 2gh} = \frac{\sqrt{2(u^2+2gh)}}{g}\sqrt{\frac{1}{2}(u^2 - 6gh)} \quad (59)$$

Corresponding to a_{max} the optimum total time of flight and optimum horizontal range are given by,

$$T_{opt} = \frac{2u}{g}\sqrt{\frac{1}{2}(1 - \frac{2gh}{u^2})} \quad (60)$$

$$R_{opt} = \frac{u^2}{g}\sqrt{(1 + \frac{2gh}{u^2})(1 - \frac{2gh}{u^2})} \quad (61)$$

If we put h=0, i.e. retaining no constraints of grazing the top of the pillars by the projectile, we get time of flight T′′ and maximum horizontal range R′′ as derived in textbooks of Dynamics:

$$T'' = \frac{2u}{\sqrt{2}\,g} \text{ and } R'' = \frac{u^2}{g} \quad (62)$$

With an angle of projection = 45°.

4. Conclusion

Four Figures 1, 2, 3, and 4 are illustrated with captions. Trajectory of the projectile launched from the platform moving forward is greater than the trajectory of the projectile launched from the platform moving backwards. The text of the paper is insightful from an academic point of view rather than its applicability in defence in that the various aspects of projectile motion discussed in this feature do not include the atmospheric resistance, which is a function of the velocity, viz, most acceptable is the square of the velocity. But an exact analytical solution cannot be worked out with atmospheric drag of square-law -velocity. However, in the defence system, cannons are fired from moving battle tanks to destroy stationary or moving targets.

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Appendix

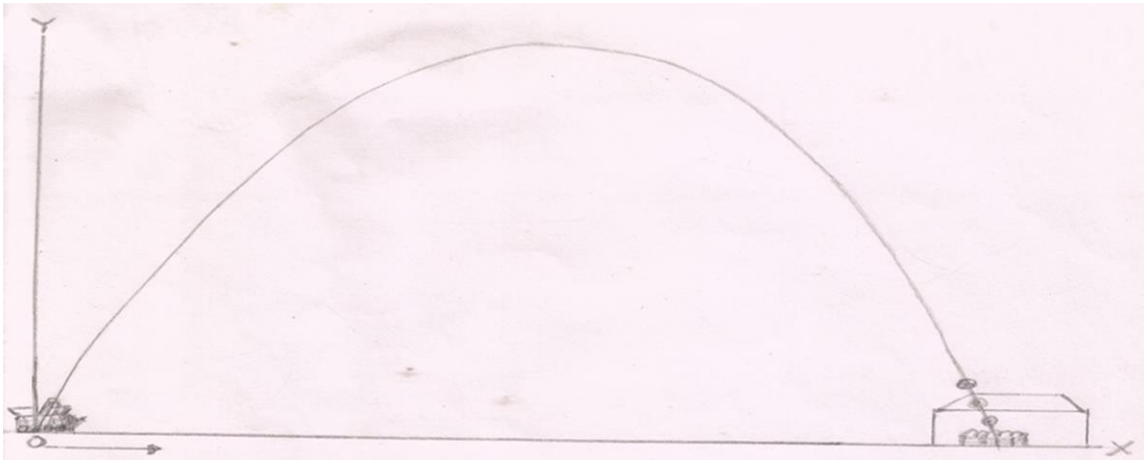

Fig. 1 Shots fired from a moving battle tank hit the sationary target: the trajectory is almost a parabole

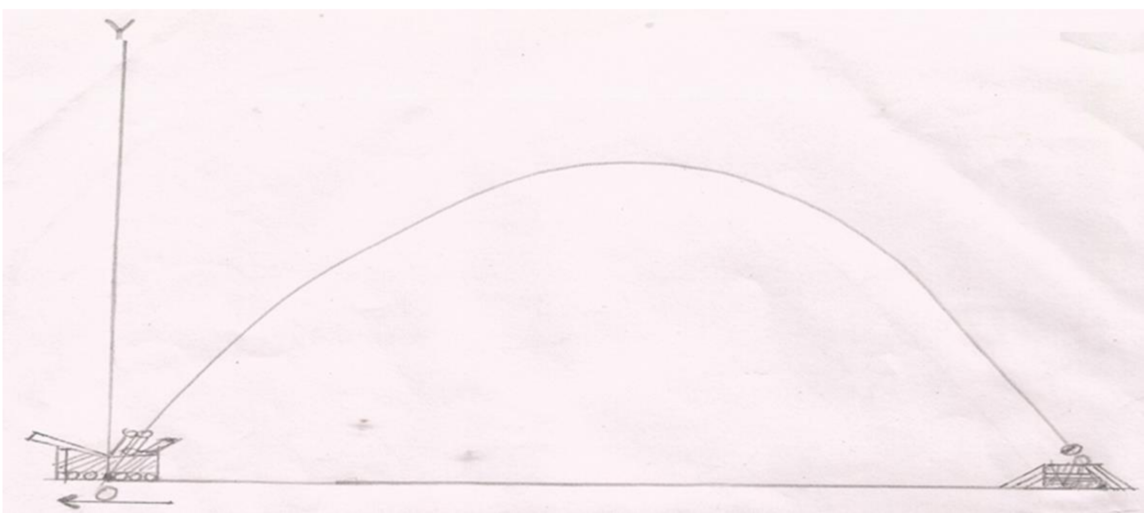

Fig. 2 Shots fired from an armoured ear moving backwards hit the sationary target: the trajectory is almost a parabole

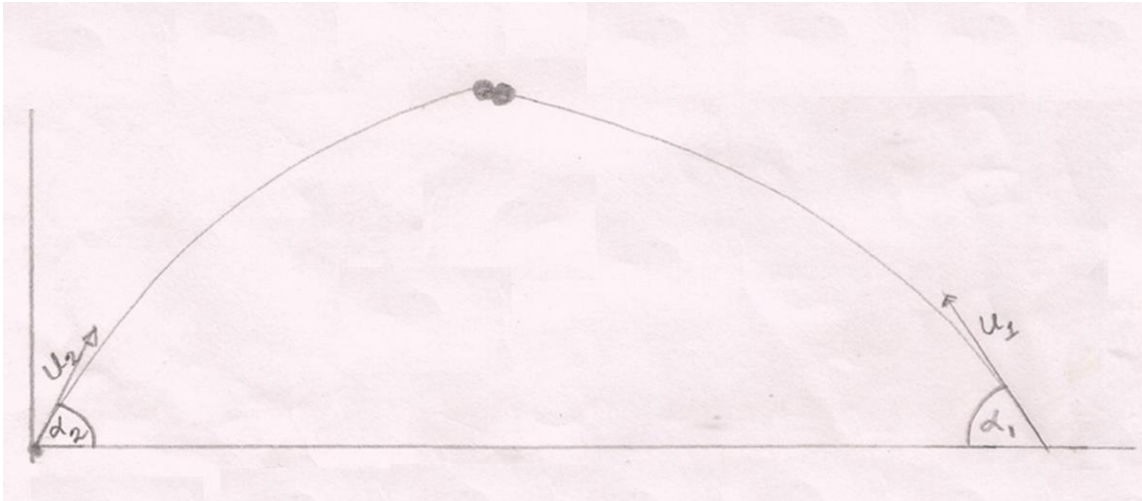

Fig. 3 Two projectiled thrown from a level platform collide with each other yielding two different trajectories

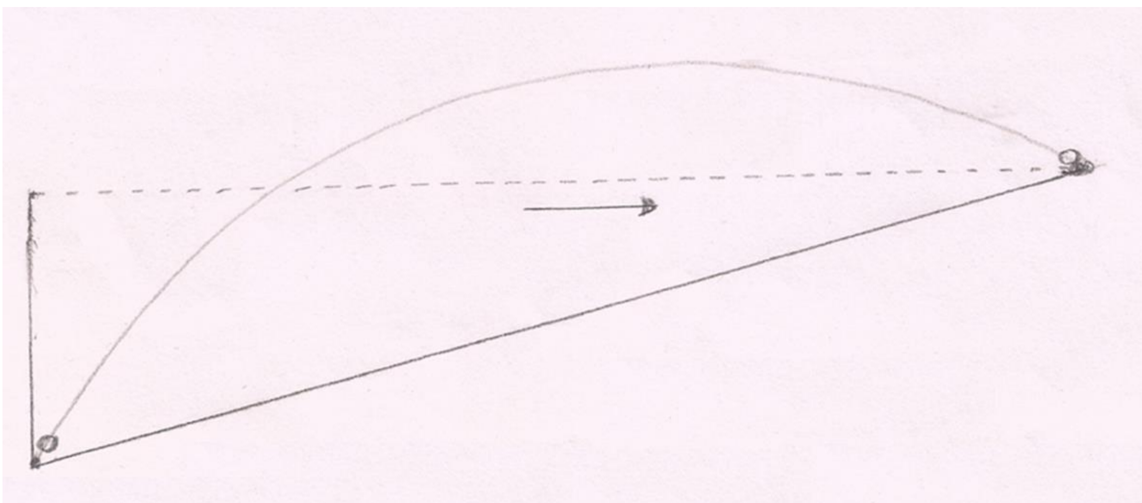

Fig. 4 A shot fired from a gun positioned at right angles to the path of a moving target strikes the latter