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Original Article

Some Typical Integrals and their Evaluations

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Abstract - Some typical integrals are innovated, and in the course of evaluating these integrals, two unusual integrals $\int\frac{dx}{1+x^4}$ and $\int\frac{dx}{x\sqrt{1+x^4}}$ are encountered. A number of such integrals are evaluated by parts. In some integrals, the integrands are inverse circular functions. In some integrals, the integrands are Logarithmic functions. Most of these integrals are converted to the foregoing integrals on simplification and elaborations to underscore the final results. Also evaluated are the integrals: $\int\frac{dx}{(1+x^2)(1+x^4)}$, $\int\frac{x^2dx}{(1+x^2)(1+x^4)}$, $\int\frac{x^3dx}{(1+x^2)(1+x^4)}$, $\int\frac{x^4dx}{(1+x^2)(1+x^4)}$, $\int\frac{x^5dx}{(1+x^2)(1+x^4)}$, $\int\frac{x^6dx}{(1+x^2)(1+x^4)}$, and $\int\frac{x^7dx}{(1+x^2)(1+x^4)}$.

Keywords - Problems, Integrals, Evaluations, Partial fractions, Definite Properties.

1. Introduction

As far as introduction is concerned, the above two vital integrals are evaluated herein without direct application of any textbook formulae. That way, eighteen integrals are innovated and evaluated mostly by parts using "dx " as the second part. The last seven integrals are evaluated by integration by partial fractions. However, this type of integrals is not found in any textbooks of Integral Calculus [1] and has not yet been published elsewhere. SN Maitra, the present author, thought of this type of integrals and evaluated them in close form.

$$\int \tan^{-1x^2} dx = x\tan^{-1x^2} - 2\int\frac{x^2dx}{1+(x^2)^2} = \tan^{-1x^2} - 2I_1 \tag{1}$$

(Integrating by parts)

Where, $I_1 = \int\frac{x^2dx}{1+x^4} = \int\frac{dx}{x^2+\frac{1}{x^2}} = \int\frac{dx}{(x+\frac{1}{x})^2-2} = \frac{1}{2}\int\{\frac{1-\frac{1}{x^2}}{(x+\frac{1}{x})^2-2} + \frac{1+\frac{1}{x^2}}{(x-\frac{1}{x})^2+2}\}dx$

$$=\frac{1}{2}\int\{\frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-2} + \frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+2}\}dx = \frac{1}{2}\{\frac{1}{2\sqrt{2}}\log\{\left(x^2-\sqrt{2}x+1\right)/\left(x^2+\sqrt{2}x+1\right)\} + \frac{1}{\sqrt{2}}\tan^{-1\frac{x^2-1}{\sqrt{2}x}}\} \tag{2}$$

1.1. Problem No. 2 and Its Solution

$$\int \tan^{-1\frac{1}{x^2}} dx = x\tan^{-1\frac{1}{x^2}} - \int\frac{x\frac{-2}{x^3}dx}{1+\frac{1}{x^4}} = x\tan^{-1\frac{1}{x^2}} + 2I_2 \tag{3}$$

Where, $I_2 = \int\frac{x^2dx}{1+x^4} = dx = I_1$

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1.2. Problem No. 3

$$\int \frac{x^2 dx}{1+x^4} \qquad \text{Vide Equation (2)}$$

$$\int \frac{dx}{x^2+x^6} = \int \frac{dx}{x^2(1+x^4)} = \int \frac{dx}{x^2} - \int \frac{x^2 dx}{1+x^4} = -\frac{1}{x} - \int \frac{x^2 dx}{1+x^4} \qquad \text{Vide Equation (2)}$$

1.3. Problem No. 4

$$\int \frac{dx}{x\sqrt{1+x^4}} = \int \frac{dx}{x^2\sqrt{x^2+\frac{1}{x^2}}} = \frac{1}{2}\int \frac{\{\left(1+\frac{1}{x^2}\right)-(1-\frac{1}{x^2})\}dx}{\sqrt{x^2+\frac{1}{x^2}}} \tag{4}$$

$$= \frac{1}{2}\int \{\frac{\left(1+\frac{1}{x^2}\right)dx}{\sqrt{(x-\frac{1}{x})^2+2}} - \frac{(1-\frac{1}{x^2})dx}{\sqrt{(x+\frac{1}{x})^2-2}}\}dx = \frac{1}{4\sqrt{2}}[\log\{x-\frac{1}{x}+\sqrt{(x-\frac{1}{x})^2+2} - \log\{x+\frac{1}{x}+\sqrt{(x+\frac{1}{x})^2-2}\}] \tag{5}$$

1.4. Problem No. 5 and Its Solution

$$\int \log(x^2+\sqrt{x^4+1})dx \qquad \text{(Integratin parts)} \tag{6}$$

$$= x\log\left(x^2+\sqrt{x^4+1}\right) - \int \frac{x\{2x+\frac{2x^3}{\sqrt{x^4+1}}}{x^2+\sqrt{x^4+1}})dx = x\log\left(x^2+\sqrt{x^4+1}\right) - 2\int \frac{x^2 dx}{\sqrt{1+x^4}}$$

$$= x\log\left(x^2+\sqrt{x^4+1}\right) - 2I_3 \tag{7}$$

Where, $$I_3 = \int \frac{x^2 dx}{\sqrt{1+x^4}} \tag{8}$$

1.5. Problem No. 6 and Its Solution

$$\int \log(x^4+1)dx \qquad \text{(Integrating by parts)}$$

$$= x\log(x^4+1) - 4\int \frac{x^4 dx}{x^4+1} = x\log(x^4+1) - 4I_4$$

Where, $$I_4 = \int \frac{x^4 dx}{x^4+1} = \int (\frac{(1+x^4)dx}{x^4+1} - \frac{dx}{x^4+1}) = x - \int \frac{dx}{x^4+1} = x - \int \frac{dx}{x^2(x^2+\frac{1}{x^2})}$$

$$= x - \frac{1}{2}\int \{\frac{1+\frac{1}{x^2}}{(x-\frac{1}{x})^2+2} - \frac{1-\frac{1}{x^2}}{(x+\frac{1}{x})^2-2}\}dx = x - \frac{1}{2}\{\frac{1}{\sqrt{2}}\tan^{-1}\frac{x^2-1}{\sqrt{2}x} - \frac{1}{2\sqrt{2}}\log\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\} \tag{9}$$

1.6. Problem No. 7 and Its Solution

$$\int x\log(x^2+\sqrt{x^4+1})dx = \frac{x^2}{2}\log\left(x^2+\sqrt{x^4+1}\right) - \int \frac{x^3 dx}{\sqrt{x^4+1}} \qquad \text{(integrating by parts)}$$

$$= \frac{x^2}{2}\log\left(x^2+\sqrt{x^4+1}\right) - -\frac{1}{2}\sqrt{x^4+1}$$

1.7. Problem No. 8 and Its Solution

$$\int x^2\log(x^2+\sqrt{x^4+1})dx = \frac{x^3}{3} - \int \frac{2x^4 dx}{3\sqrt{x^4+1}} = \frac{x^3}{3} - \frac{2}{3}\int (\sqrt{x^4+1} - \frac{1}{\sqrt{x^4+1}})dx$$

1.8. Problem No. 9 and Its Solution

$$\int \frac{\tan^{-1}x^2}{x^2}dx = -\frac{\tan^{-1}x^2}{x} + 2\int \frac{dx}{1+x^4} \qquad \text{(Integrating by parts)}$$

$$=-\frac{\tan^{-1x^2}}{x}+2I_0 \tag{10}$$

$I_0=\int\frac{dx}{1+x^4}$ (Vide Equation (9))

1.9. Problem No. 10 and Its Solution

$$\int\frac{\log(1+x^4)dx}{x^2}=-\frac{1}{x}\log(1+x^4)+4\int\frac{x^2dx}{1+x^4}$$ (Vide Equation (2))

1.10. Problem No. 11

$$\int\tan^{-1\sqrt{x}}dx=x\tan^{-1\sqrt{x}}-\int x\frac{1}{2\sqrt{x}}\frac{dx}{1+x}$$ (Integrating by parts)

(putting $x=z^2$ so that $dx=2zdz$ in the integral part)

$$\int\frac{z^2dz}{1+z^2}=\int(1-\frac{dz}{1+z^2})dz=z-\tan^{-1z}=\sqrt{x}-\tan^{-1}\sqrt{x} \tag{11}$$

1.11. Problem No. 12 and Its Solution

$$\int\frac{\log(1+x^4)dx}{x^3}=-\frac{\log(1+x^4)}{2x^2}+\int\frac{4x^3dx}{2x^2(1+x^4)}$$ (Integrating by parts)

$$=-\frac{\log(1+x^4)}{2x^2}+\int\frac{2xdx}{(1+x^4)}=-=-\frac{\log(1+x^4)}{2x^2}+\int\frac{dx^2}{(1+x^4)}=-=-\frac{\log(1+x^4)}{2x^2}+\tan^{-1x^2} \tag{12}$$

1.12. Problem No. 13 and Its Solution

$$\int\frac{dx}{(1+x^2)(1+x^4)} \tag{13}$$

Separating the integrand by partial fractions,

$$\frac{1}{(1+x^2)(1+x^4)}=\frac{A}{(1+x^2)}+\frac{Bx^2+C}{(1+x^4)} \tag{14}$$

Or, $1=A(1+x^4)+(Bx^2+C)(1+x^2)$

Comparing the constant and coefficients of x^2 and x^2 are obtained.

A+C=1, B+C=A+B=0, which leads to

$$A=C=\frac{1}{2}, \text{ and } B=-\frac{1}{2} \tag{15}$$

Equations (13), (14) and (15) conform to,

$$\int\frac{dx}{(1+x^2)(1+x^4)}=\int\{\frac{1}{2(1+x^2)}+\frac{1-x^2}{2(1+x^4)}\}dx=\frac{1}{2}\tan^{-1x}-\int\frac{\left(1-\frac{1}{x^2}\right)dx}{2\{\left(x+\frac{1}{x}\right)^2-2\}\big)}$$

$$=\frac{1}{2}\tan^{-1x}-\frac{1}{4\sqrt{2}}\log\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} \tag{16}$$

1.13. Problem No. 14 and Its Solution

$$=\int\frac{xdx}{(1+x^2)(1+x^4)} \tag{17}$$

(putting $y=x^2$ so that $dy=2xdx$)

$$=\frac{1}{2}\int\frac{dy}{(1+y)(1+y^2)}=\frac{1}{2}\int\{\frac{1}{(1+y)}+\int\frac{(1-y)}{(1+y^2)}\}dy$$

$$=\frac{1}{2}\{\log(1+y)--\frac{1}{2}\log(1+y^2)+\tan^{-1y}\}$$

$$=\frac{1}{2}\{\log(1+x^2)-\frac{1}{2}\log(1+x^4)+\tan^{-1x^2}\}$$

1.14. Problem No. 15 and Its Solution

$$\int\frac{x^2dx}{(1+x^2)(1+x^4)}=\int\left\{\frac{-1}{2(1+x^2)}+\frac{1+x^2}{2(1+x^4)}\right\}dx \quad (18)$$

$$=\frac{1}{2}\{-\tan^{-1x}+\int\frac{\left(1+\frac{1}{x^2}\right)dx}{\left(\left(x-\frac{1}{x}\right)^2+2\right)}\}$$

$$=\frac{1}{2}\{-\tan^{-1x}+\frac{1}{\sqrt{2}}\tan^{-1}\frac{x-\frac{1}{x}}{\sqrt{2}}\}$$

$$=\int\frac{x^3dx}{(1+x^2)(1+x^4)} \qquad \text{(Multiplying (18) under integral sign by x)} \quad (19)$$

$$=\int\left\{\frac{-x}{2(1+x^2)}+\frac{x+x^3}{2(1+x^4)}\right\}dx \qquad \text{(As done earlier)}$$

$$=-\frac{1}{4}\log(1+x^2)+\frac{1}{4}\tan^{-1x^2}+\frac{1}{8}\log(1+x^4) \quad (20)$$

$$\int\frac{x^4dx}{(1+x^2)(1+x^4)}=\int\{\frac{1}{2(1+x^2)}-\frac{1-x^2}{2(1+x^4)}\}dx$$

$$=\frac{1}{2}\tan^{-1x}-\int\frac{1-x^2}{2(1+x^4)}dx$$

$$=\frac{1}{2}\{\tan^{-1x}+\int\frac{-\frac{1}{x^2}+1}{\left(\frac{1}{x}+x\right)^2-2}dx\}$$

$$=\frac{1}{2}\{\tan^{-1x}+\frac{1}{2\sqrt{2}}\log\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\} \quad (21)$$

1.15. Problem No. 16 and Its Solution

$$\int\frac{x^5dx}{(1+x^2)(1+x^4)} \quad (22)$$

$$=\int\{\frac{x}{2(1+x^2)}-\frac{x-x^3}{2(1+x^4)}\}dx$$

$$=\frac{1}{4}\log((1+x^2)-\frac{1}{4}\tan^{-1x^2}+\frac{1}{8}\log((1+x^4) \quad (23)$$

1.16. Problem No. 17 and Its Solution

$$\int\frac{x^6dx}{(1+x^2)(1+x^4)}=\int\{\frac{x^2}{2(1+x^2)}-\frac{x^2-x^4}{2(1+x^4)}\}dx$$

$$=\frac{1}{2}\int\{1-\frac{1}{(1+x^2)}-\frac{x^2+1-(1+x^4)}{(1+x^4)}\}dx$$

$$=\frac{1}{2}\int\{1-\frac{1}{(1+x^2)}-\frac{x^2+1}{(1+x^4)}+1\}dx$$

$$=\frac{1}{2}\int\{1-\frac{1}{(1+x^2)}+\frac{-\left(\frac{1}{x}\right)^2-1}{\left(\frac{1}{x}-x)^2+2\right)}+1\}dx$$

$$=\frac{1}{2}(2x-\tan^{-1x}+\frac{1}{\sqrt{2}}\tan^{-1\frac{1-x^2}{x\sqrt{2}}})$$

1.17. Problem No. 18 and Its Solution

$$\int\frac{x^7dx}{(1+x^2)(1+x^4)}=\int\{\frac{x^3}{2(1+x^2)}-\frac{x^3-x^5}{2(1+x^4)}\}\,dx \qquad \text{(Multiplying integrand of problem No 18)}$$

Here, we need to evaluate the first integral of the right-hand side, whereas the remaining integrals are available from (20) and (22).

$$\int\{\frac{x^3dx}{2(1+x^2)}=\frac{1}{2}\int\{x-\frac{x}{1+x^2})dx$$

$$=\frac{1}{4}\{x^2-\log(1+x^2)\}$$

2. Conclusion

It is speculated that harder integrals that can be evaluated in close form can be innovated. Nevertheless, such integrals will hopefully serve as brainstorming exercises for students, teachers, and researchers in mathematics education. After attempting many trials in the foregoing analysis, It is confirmed that integrals $\int\frac{dx}{\sqrt{1+x^4}}$ and $\int\frac{x^2dx}{\sqrt{1+x^4}}$ can not be evaluated in close form. However, in case this type of integral appears while tackling any realworld problems, we need to resort to Binomial expansions if x<1 and if x>1, then $\frac{1}{x}<1.$ Thereafter, term – by – term integrations are carried out till acceptable approximate results are acquired. Further, it is interesting to observe that $\int\sqrt{\tan\theta}\,d\theta$ and $\int\sqrt{\cot\theta}\,d\theta$ can respectively be reduced to the forms $2\int\frac{x^2dx}{1+x^4}$ and $2\int\frac{dx}{1+x^4}$ by substituting $\tan\theta=x^2$.

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