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Original Article

Some Unusual Integrals and their Evaluation

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Abstract - The first integral that is available in textbooks of Integral Calculus in definite integral form is evaluated in close form. Thereafter, the upper and lower limits are assigned to the integral, which is evaluated obviously without the use of the properties of the definite integral. The process of integration calls for change of the variable by substitution and separation of the integrand into partial fractions. The second integral is evaluated in close form and can not be evaluated in definite integral form by use of properties of the definite integral. Some more integrals that have not yet found a home in the literature are also evaluated.

Keywords - Problems, Integrals, Evaluations, Partial fractions, Definite Properties

1. Introduction

In almost all Integral Calculus books [1, 2], some available integrands are found to be circular functions, such as $\int \sin^3 x\cos^4 xdx$, $\int \sin^4 x\cos^4 xdx$, $\int \sec^4 x\tan^4 xdx$, $\int \frac{dx}{a+bsinx}$, $\int \frac{sec^2xdx}{a+btanx}$, $\int \frac{sec^4xdx}{a+btanx}$ etc which can be evaluated using the usual formulae. Herein are mainly solved two integral problems that are not yet found in any Calculus books or elsewhere.

2. Problem No. 1 Evaluate $\int \frac{sin^3xdx}{sin^3x+cos^3x}$

$$I=\int \frac{sin^3xdx}{sin^3x+cos^3x} = \int \frac{tan^3xdx}{tan^3x+1} = \int sec^2x \frac{tan^3xdx}{sec^2x(tan^3x+1)} = \int \frac{tan^3xd(tanx)}{(1+tan^2x)(tan^3x+1)} \quad (1)$$

The integrand is now separated into partial fractions after putting tanx=y,

$$\text{Then the integrand} = \frac{\tan^3 x}{(1+\tan^2 x)(\tan^3 x+1)} = \frac{y^3}{(1+y^2)(y^3+1)} = \frac{y^3}{(1+y^2)(y+1)(y^2-y+1)}$$

$$=\frac{A+By}{1+y^2}+\frac{C}{1+y}+\frac{Dy+E}{y^2-y+1} \text{ (By rules of separation into partial fractions)} \quad (2)$$

Where, A, B, C, D are constants to be evaluated by comparison method:

$$(A+By)(1-y+y^2)(1+y)+C(1+y^2)(y^2-y+1)+(Dy+E)(1+y^2)(1+y)=y^3 \quad (3)$$

$$B+C+D=0 \quad (4)$$

$$A-C+D+E=1 \quad (5)$$

$$A-B+2C+D+E=0 \tag{6}$$

$$-A+B-C+D+E=0 \tag{7}$$

$$A+C+E=0 \tag{8}$$

Combining (7) and (6),

$$C+2(D+E)=0 \tag{9}$$

Combining (8) and (9),

$$A-2D-E=0 \tag{10}$$

$$2A-2B+3C=0 \tag{11}$$

Combining (7) and (8),

$$B+D+2E=0 \tag{12}$$

Combining (4) and (12),

$$E=\frac{C}{2} \tag{13}$$

Combining (8) and (13),

$$A=\frac{-3C}{2} \tag{14}$$

Combining (11) and (14),

$$B=0 \tag{15}$$

Combining (13) and (15) with (12),

$$D=-C \tag{16}$$

Using (13) to (16) in (5) is obtained.

$$A-C+D+E=1$$

$$\text{Or, } \frac{-3C}{2}-C-C+\frac{C}{2}=1 \qquad \text{Or, } \frac{-3C-4C+C}{2}=1$$

$$C=-\frac{1}{3} \tag{17}$$

$$E=-\frac{1}{6}$$

Using (17) in (16) and (14) respectively,

$$D=\frac{1}{3} \tag{18}$$

$A=\frac{1}{2}$ (19)

Substituting in (2) the numerical values of the constants as delineated above.

The integrand= $\frac{A+By}{1+y^2}+\frac{C}{1+y}+\frac{Dy+E}{y^2-y+1}$

$= \frac{1}{2(1+y^2)}-\frac{1}{3(1+y)}+\frac{2y-1}{6(y^2-y+1)}$ (20)

Integrating (20) with respect to y is obtained.

$I=\frac{1}{2}tan^{-1}y-\frac{1}{3}\log(1+y)+\frac{1}{6}\log(y^2-y+1) =\frac{1}{2}tan^{-1}y-\frac{1}{6}\log(1+y)^2+\frac{1}{6}\log(y^2-y+1)$

$=\frac{1}{2}tan^{-1}y+\frac{1}{6}\log\frac{y^2-y+1)}{y^2+2y+1)}$ + Constant of integration (21)

Now find the concerned definite integral,

$J=\int_0^{\frac{\pi}{2}}\frac{sin^3xdx}{sin^3x+cos^3x}$ (22)

As x varies from 0 to $\frac{\pi}{2}$, y=tanx varies from 0 to ∞.

Hence (22) and (21) give,

$J=\frac{1}{2}\left(tan^{-1}\infty-\frac{1}{2}tan^{-1}0\right)+\frac{1}{6}(Lim_{y\to\infty}\log\frac{y^2-y+1)}{y^2+2y+1)}-log1)$

Or, $J=\frac{\pi}{4}$ (23)

3. Problem No. 2 Evaluate $\int\frac{dx}{1+sin^3x}$

$I=\int\frac{dx}{1+sin^3x}=\int\frac{dx}{(1+snx)(1-sinx+sin^2x)}=\int\frac{dx}{\left(1+2sin\frac{x}{2}cos\frac{x}{2}\right)\{1-2sin\frac{x}{2}cos\frac{x}{2}+(2sin\frac{x}{2}cos\frac{x}{2})^2\}}$

$=\int\frac{sec^2\frac{x}{2}sec^2\frac{x}{2}dx}{\left(1+tan\frac{x}{2}\right)^2\{1-2tan\frac{x}{2}+5tan^2\frac{x}{2}\}}$ (putting $\tan\frac{x}{2}=y$ so that $2y=\sec^2\frac{x}{2}dx$)

$I=\int\frac{2(1+\tan^2\frac{x}{2})d(\tan\frac{x}{2})}{\left(1+\tan\frac{x}{2}\right)^2\{1-2\tan\frac{x}{2}+5\tan^2\frac{x}{2}\}}=\int\frac{2(1+y^2)dy}{(1+y)^{2\{}1-2y+5y^2)}$ (24)

The above integrand is separated into partial fractions:

$\frac{(1+y^2)}{(1+y)^2(1-2y+5y^2)}=\frac{A}{1+y}+\frac{B}{(1+y)^2}+\frac{C+Dy}{1-2y+5y^2}$ (25)

Or, $1+y^2=A(1+y)(1-2y+5y^2)+B(1-2y+5y^2)+(C+Dy)(1+y)^2$ (26)

Comparing the coefficients of y^3, y^2, y and constants, are obtained.

5A+D=0 (27)

$3A+5B+C+2D=1$ (28)

$-A-2B+2C+D=0$ (29)

$A+B+C=1$ (30)

From (27), $D=-5A$ (31)

Adding (29) and (30),

$-B+3C+D=1$ (32)

Using (31) and (29),

$-6A-2B+2C=0$

Or, $-B+C=3A$ (33)

Similarly,

$2C=1+2A$ (34)

Or, $C=A+\frac{1}{2}$ (35)

$B=1-A-C=1-A-(+A+\frac{1}{2})=-2A+\frac{1}{2}$ (36)

$D=-5A$ (37)

Thus, constants B, C, D are expressed in terms of A and are substituted in (28) to yield,

$3A+5(-2A+\frac{1}{2})+A+\frac{1}{2}-10A=1$

Or, $A=\frac{1}{8}$, $B=\frac{1}{4}$, $C=\frac{5}{8}$, $D=-\frac{5}{8}$ (38)

Putting the above values of the constants in the integrand (25),

$$\frac{(1+y^2)}{(1+y)^2(1-2y+5y^2)}=\frac{1}{8(1+y)}+\frac{1}{4(1+y)^2}+\frac{5(1-y)}{8(1-2y+5y^2)} \quad (39)$$

In view of (39) integral (24) turns out to be,

$$I=\int\frac{2(1+y^2)dy}{(1+y)^2\{1-2y+5y^2)}=2\int\{\frac{1}{8(1+y)}+\frac{1}{4(1+y)^2}+\frac{5(1-y)}{8(1-2y+5y^2)}\}dy=\frac{1}{4}\log(1+y)-\frac{1}{2(1+y)}-\frac{1}{8}\log(1-2y+5y^2)$$

As a consequence of this, the integral I becomes without writing the constant of integration,

$$I=\frac{1}{4}\log(1+y)-\frac{1}{2(1+y)}-\frac{1}{8}\log(1-2y+5y^2)$$

$$I=\frac{1}{8}\log\frac{1+2y+y^2}{1-2y+5y^2}-\frac{1}{2(1+y)} \quad (40)$$

The integral when x varies from 0 to $\frac{\pi}{2}$ ie y varies from 0 to ∞ *is* given by,

$$J=\frac{1}{4} \tag{41}$$

4. Problem No. 3 Evaluate $\int \frac{dx}{sinx+sin^3x}$

$I=\int \frac{dx}{sinx(1+sin^2x)}$ (Multiplying num and deno by sinx)

$=\int \frac{sinxdx}{sin^2x(1+sin^2x)}=\int \frac{-d(cosx)}{(1-cos^2x)(2-cos^2x)} =\int\{\frac{1}{(2-cos^2x)} - \frac{1}{(1-cos^2x)}\}$ d(cosx)

$=\frac{1}{2\sqrt{2}} log \frac{\sqrt{2}+cosx}{\sqrt{2}-cosx}$-log$\frac{1+cosx}{1-cosx}$ +constant of integration

5. Problem No. 4 Evaluate $\int \frac{dx}{sin^2x+sin^3x}$

$I = \int \frac{dx}{sin^2x+sin^3x} =\int \frac{dx}{sin^2x(1+sinx)}$ (Multiplying num and deno by (1-sinx)

$=\int \frac{(1-sinx)dx}{sin^2x(1-sin^2x)}=\int \frac{(1-sinx)dx}{sin^2xcos^2x} =\int\{4cosec^2 2xdx + \frac{d(cosx)}{(1-cos^2x)cos^2x}\}$

$=-2cot2x+\int\{\frac{d(cosx)}{(1-cos^2x)}+\frac{d(cosx)}{cos^2x}\} =-cot2x+log\frac{1+cosx}{1-cosx} - secx$ +const.

6. Conclusion

Definite integral (22) is available in almost all Integral Calculus books, and its value can be found in a simple way by applying the properties of definite integrals. However, this is also showcased here, leading to the same result (23). Proper adjustments and substitutions derive the other integrals.

$$J = \int_0^{\frac{\pi}{2}} \frac{sin^3xdx}{sin^3x+cos^3x} = \int_0^{\frac{\pi}{2}} \frac{sin^3(\frac{\pi}{2}-x)dx}{sin^3(\frac{\pi}{2}-x)+cos^3(\frac{\pi}{2}-x)}$$

$$J = \int_0^{\frac{\pi}{2}} \frac{cos^3xdx}{cos^3x+sin^3x}$$

Adding, $2J=\int_0^{\frac{\pi}{2}} \frac{(sin^3x+cos^3x)dx}{sin^3x+cos^3x}=\frac{\pi}{2}$

$$J=\frac{\pi}{4}$$

References

[1] S. Narayanan, and T.K. Manicavachagom Pillay, *Calculus: Volume 2*, S. Viswanathan Printers & Publishers Pvt Ltd, 1996.
[2] R. Bharadwaj, *Mathematics Guidelines with Formulae and Definitions*, Computech Publication Ltd. New Asian, New Delhi, 1988.