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Original Article

Launch of a Rocket: Rectilinear Climbing Powered Flight in Atmosphere Followed by Coasting

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Abstract - *A Rocket/missile is constrained to fly in the atmosphere along an oblique path under the action of variable thrust and drag along the tangent to the flight path and gravitational force, whereas the component of the gravitational force balances the lift normal to the flight path at all time instants during the powered flight. At the end of the propellant consumption, the rocket travels freely as a projectile in the atmosphere. The governing differential equations of motion are set up and solved by applying the relevant initial conditions to determine the overall horizontal range due to the powered flight followed by the coasting flight*.

Keywords - *Flight path, Velocity, Horizontal distance, Gravitational force.*

1. Introduction

In warfare, it is observed that a rocket/missile is fired to zoom vertically upwards or to travel vigorously along an oblique rectilinear path followed by its curvilinear motion. Angelo Miele1dealt with short-range air-to-air missiles flown assuming negligible the component on the tangent to the flight and with sounding rocket in the atmosphere. He also incorporated the approximate performance of rocket-powered aircraft operating with constant propellant mass flow and computed the propellant mass ratios required to perform certain typical manuevers, taking into consideration the atmospheric resistance and fuel consumption. He considered climbing flights with constant velocity as well as with constant dynamic pressure. S.N. Maitra [2] published several papers, each pertaining to a variable thrust.

2. Rocket Launched to Fly in a Rectilinear Path

Let a powered rocket of lift off mass m_0 be launched at an angle α to the horizontal and fly in a rectilinear path of inclination α to the horizontal by manipulating its mass variation vis-à-vis creation of thrust till the entire propellant is consumed, and after that it nurtures free flight to fall on the horizontal ground. In fact, it attains a height and descends entailing projectile motion.

3. Equations of Motion of Powered Flight in the Atmosphere

Let $m_p$be the propellant mass consumed after attaining a velocity u at a height h, L the lift, D the drag, C_D the drag coefficient, C_L the lift coefficient, S the reference area and ρ the air density. During the rectilinear path of the rocket, the lift normally acts to the straight line path and is neutralised by the component of the weight of the rocket in that direction to bring forth rectilinear flight. If v is the velocity and m the mass of the rocket at any instant of time t during the rectilinear flight,

$$D=\frac{1}{2}\rho C_D S v^2 \tag{1}$$

$$L=\frac{1}{2}\rho C_L S v^2 = mg\cos\alpha \tag{2}$$

$$\text{Or,} v^2 = \frac{2mg\cos\alpha}{\rho C_L}$$

$$\text{Or,} v=\sqrt{\frac{2mg\cos\alpha}{\rho C_L S}} \tag{3}$$

and the initial velocity to be generated is,

$$v_0=\sqrt{\frac{2m_0 g\cos\alpha}{\rho C_L S}} \tag{4}$$

It is worthwhile to note that in this feature, the rocket does not start from rest but starts with a large initial velocity generated by an impulsive action caused by burning a quantity of fuel in the quickest time. If m' is the propellant burnt to impart the rocket a large velocity in the quickest time, the equation of motion that holds is,

$$\frac{mdv}{dt} = -V_E' \frac{dm}{dt} \tag{5}$$

$$\text{Or,} v_0 = V_E' \log\frac{m_0}{m_0 - m'} \tag{6}$$

Where, V_E'=velocity of the burning gas exhausted through the nozzles at the rear of the rocket in a very short time such that $V_E' >> V_E$. Equation (3) suggests that as the mass of the rocket decreases due to fuel consumption, its velocity decreases.

Differentiating (3) with respect to time t,

$$\frac{dv}{dt}=\sqrt{\frac{g\cos\alpha}{2\rho C_L S}}\frac{1}{\sqrt{m}}\frac{dm}{dt} \tag{7}$$

The forces acting along the tangent to the flight path are,

Drag D, [vide Equation (1)]

$$\text{Thrust} = -V_E \frac{dm}{dt},$$

V_E = exhaust velocity of the rocket.

The component of the gravitational force $= mg\sin\alpha$ (8)

In the light of the above relationships, the equations of motion of the rocket are set up :

$$m\frac{dv}{dt} = -V_E \frac{dm}{dt} - D - mg\sin\alpha \tag{9}$$

In view of (1),

$$m\frac{dv}{dt} = -V_E \frac{dm}{dt} - \frac{1}{2}\rho C_D S v^2 - mg\sin\alpha$$

Eliminating v and v^2 from this Equation by use of (3) and (2) and denoting,

$$\mu = \frac{C_D}{C_L} \text{ and } k = \sqrt{\frac{g\cos\alpha}{2\rho C_L S}} \tag{10}$$

One gets,

$$\sqrt{\frac{g\cos\alpha}{2\rho C_L S}} \frac{m}{\sqrt{m}} \frac{dm}{dt} = -V_E \frac{dm}{dt} - C_D \frac{mg\cos\alpha}{C_L} - mg\sin\alpha$$

$$\text{Or,} \frac{km}{\sqrt{m}} \frac{dm}{dt} = -V_E \frac{dm}{dt} - \mu mg\cos\alpha - mg\sin\alpha$$

$$\text{Or,-} \frac{km}{\sqrt{m}} + V_E) \frac{dm}{dt} = \mu mg\cos\alpha + mg\sin\alpha \tag{11}$$

3.1. Solutions to the Problem

With the initial conditions: At t=o,m=$m_{0,}$v=$v_{0,}$ $x = 0$, (12)

Integrating (11), the mass at any instant of time t is given by,

$$2k(\sqrt{m_0} - \sqrt{m}) + V_E \log\frac{m_0}{m} = g(\mu\cos\alpha + \sin\alpha)t$$

$$t = \frac{2k(\sqrt{m_0} - \sqrt{m}) + V_E \log\frac{m_0}{m}}{g(\mu\cos\alpha + \sin\alpha)} \tag{13}$$

Employing (3)), we get

$$v = \frac{dx}{dt} = \sqrt{\frac{2mg\cos\alpha}{\rho C_L S}} \tag{14}$$

Eliminating t from this equation, employing the values of k and μ and choosing m as the independent variable by means of (10) and (11) subject to the initial conditions(12) is obtained the distance x travelled by the rocket along the inclined path :

$$\frac{dx}{dm} = -\sqrt{m} \sqrt{\frac{2g\cos\alpha}{\rho C_L S}} . \frac{(\frac{k}{\sqrt{m}} + \frac{V_E}{m})}{g(\mu\cos\alpha + \sin\alpha)} \tag{15}$$

Integrating (15) using the initial conditions as above,

$$\text{Or, } X = 2k \frac{k(m_0 - m) + 2V_E(\sqrt{m_0} - \sqrt{m})}{g(\mu\cos\alpha + \sin\alpha)} \tag{16}$$

If T is the time taken to exhaust the entire propellant m_p and the distance X travelled because of (13)and (16):

$$T = \frac{2k(\sqrt{m_0} - \sqrt{m_0 - m_p}) + V_E \log\frac{m_0}{m_0 - m_p}}{g(\mu\cos\alpha + \sin\alpha)} \tag{17}$$

$$X = 2k \frac{k(m_0 - m_p) + 2V_E(\sqrt{m_0} - \sqrt{m_0 - m_p})}{g(\mu\cos\alpha + \sin\alpha)} \tag{18}$$

Using (3) and(18), the velocity u acquired, the horizontal distance $x_1$covered and height y_1 attained at the end of the propellant consumption are given by,

$$u=\sqrt{\frac{2(m_0-m_p)g\cos\alpha}{\rho C_L S}} \tag{19}$$

$$x_1=2k\frac{k(m_0-m_p)+2V_E(\sqrt{m_0}-\sqrt{m_0-m_p})}{g(\mu\cos\alpha+\sin\alpha)}\cos\alpha \tag{20}$$

$$y_1=2k\frac{k(m_0-m_p)+2V_E(\sqrt{m_0}-\sqrt{m_0-m_p})}{g(\mu\cos\alpha+\sin\alpha)}\sin\alpha \tag{21}$$

At the end of the propellant consumption, the rocket is exposed to simple projectile motion, neglecting the atmospheric resistance as if the rocket is a projectile projected at angle α to the horizontal with velocity u leading to the formation of the equations of motion including the horizontal distance x_2 described after falling on the ground descending a height $y_1 in$ time t_1:

$$-y_1=u\sin\alpha t_1-\frac{1}{2}gt_1^2 \tag{22}$$

$$x_2 = u\cos\alpha t_1 \tag{23}$$

Eliminating α between (22) and (23),

$$(ut_1)^2 = x_2^2 + (-y_1 + \frac{1}{2}gt_1^2)^2 \tag{24}$$

Or, $(\frac{1}{2}gt_1^2)^2 - (u^2 + y_1 g)t_1^2 + x_2^2 + y_1^2 = 0$

Or, $t_1^2 = \frac{(u^2+y_1 g)+\sqrt{(u^2+y_1 g)^2-g^2(x_2^2+y_1^2)}}{2\,g^2}$

$$\text{Or, } t_1 = \sqrt{\frac{(u^2+y_1 g)+\sqrt{(u^2+y_1 g)^2-g^2(x_2^2+y_1^2)}}{2\,g^2}} \tag{25}$$

Gives the duration of the coasting flight to cover the horizontal range x_2 and the angle α of projection is obtained by use of (23) and (25), given the initial velocity u and height y_1.

$$\cos\alpha = \frac{x_2}{ut_1} = \frac{x_2}{u\sqrt{\frac{(u^2+y_1 g)+\sqrt{(u^2+y_1 g)^2-g^2(x_2^2+y_1^2)}}{2\,g^2}}}$$

Solving quadratic Equation (22) is obtained at the time of descending flight with initial velocity u and angle α of projection to the horizontal:

$$t_1 = \frac{u\sin\alpha+\sqrt{u^2\sin^2\alpha+2gy_1}}{g} \tag{26}$$

Combining (17) and (26), the total time T' taken by the rocket to fall on the ground is given by,

$$T'=T+t_1 = \frac{2k(\sqrt{m_0}-\sqrt{m_0-m_p})+V_E\log\frac{m_0}{m_0-m_p}}{g(\mu\cos\alpha+\sin\alpha)}+\frac{u\sin\alpha+\sqrt{u^2\sin^2\alpha+2gy_1}}{g}$$

4. Maximum Coasting Range and Optimum Time of Flight

From (24), the maximum coasting horizontal range is given by,

$$\frac{dx_2}{d(t_1^2)} = u^2 - g\left(\frac{1}{2}gt_1^2 - y_1\right) = 0 \tag{27}$$

$$\text{Or,} t_1^2{}_{opt} = 2\left(\frac{u^2+gy_1}{g^2}\right) \tag{28}$$

$$t_{1(opt)} = \sqrt{2\left(\frac{u^2+gy_1}{g^2}\right)} \tag{29}$$

Using (2) and (28) in (24) is obtained the maximum horizontal range x_{2max} for coasting flights:

$$(x_2^2)max = (ut_1)^2 - (-y_1 + \frac{1}{2}gt_1^2)^2$$

$$= 2u^2\left(\frac{u^2+gy_1}{g^2}\right) - (\frac{u^2}{g})^2$$

$$\text{Or, } (x_2^2)max = \frac{u^4}{g^2} + \frac{2u^2y_1}{g} \tag{30}$$

$$x_{2max} = \sqrt{\frac{u^4}{g^2} + \frac{2u^2y_1}{g}}$$

Using (29) and(30),

$$\cos\alpha_{0pt} = \frac{x_{2max}}{ut_{0pt}} = \frac{\sqrt{\frac{u^4}{g^2}+\frac{2u^2y_1}{g}}}{\sqrt{2\left(\frac{u^4+gu^2y_1}{g^2}\right)}} \tag{31}$$

$$\sin\alpha_{0pt} = \frac{\frac{u^2}{g}}{\sqrt{2\left(\frac{u^4+gu^2y_1}{g^2}\right)}} \tag{32}$$

$$\tan\alpha_{0pt} = \frac{\frac{u^2}{g}}{\sqrt{\frac{u^4}{g^2}+\frac{2u^2y_1}{g}}} \tag{33}$$

Making use of (23) and (25),

$$x_2 = u\cos\alpha\frac{u\sin\alpha+\sqrt{u^2\sin^2\alpha+2gy_1}}{g} \tag{34}$$

Combining (20) and (34), the overall horizontal range turns out to be,

$$R = x_1 + x_2 = \frac{\frac{g\cos\alpha}{\rho C_L S}(m_0-m_p)+2\sqrt{\frac{2g\cos\alpha}{\rho C_L S}}V_E(\sqrt{m_0}-\sqrt{m_0-m_p})}{g(\mu\cos\alpha+\sin\alpha)}\cos\alpha + \frac{u\sin\alpha+\sqrt{u^2\sin^2\alpha+2gy_1}}{g}u\cos\alpha \tag{35}$$

For a maximum overall range from (35),

$$\frac{dR}{d\alpha} = \left[\frac{\{-\frac{g\sin2\alpha}{\rho C_L S}(m_0-m_p)-3\sin\alpha\sqrt{\frac{2g\cos\alpha}{\rho C_L S}}V_E(\sqrt{m_0}-\sqrt{m_0-m_p})\}}{g(\mu\cos\alpha+\sin\alpha)} + \frac{\left\{\frac{g\cos\alpha}{\rho C_L S}(m_0-m_p)+2\sqrt{\frac{2g\cos\alpha}{\rho C_L S}}V_E(\sqrt{m_0}-\sqrt{m_0-m_p})\right\}(-\mu\sin\alpha+\cos\alpha)\cos\alpha}{g(\mu\cos\alpha+\sin\alpha)^2}\right] + \frac{u^2\cos2\alpha + \frac{u^4\sin2\alpha\cos2\alpha - gu^2y_1\sin2\alpha}{2\sqrt{u^4\sin^2\alpha\cos^2\alpha+2u^2gy_1\cos^2\alpha}}}{g} = 0 \tag{36}$$

Value of α, though a herculean task is obtainable from (36) by the numerical process; it is substituted in (35) to compute the maximum total range R_{max}.

5. The Projectile Motion with Atmospheric Resistance Proportional to the Velocity and its Solution

In the above feature, we have totally neglected the atmospheric resistance during the coasting flight at most from an academic point of view, so much so that permissible square-law-velocity drag does not admit of analytical solution. However, because of the high velocity of the rocket, the accuracy of the solution can be significantly improved by incorporating the drag proportional to the velocity.

In this context, the differential equations of motion are formed as,

$$\frac{du\prime}{dt} = \frac{d^2x}{dt^2} = -cu' \quad (37)$$

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = -cv - g \quad (38)$$

Where, u$\cos\alpha$ and u$\sin\alpha$ are the initial horizontal and vertical components of the velocity; x and y are the horizontal distance traversed, and height descended attaining horizontal velocity u' and vertical v at any instant of time t reckoned from the beginning of the coasting flight; g is the constant acceleration due to gravity, c the constant of proportionality;

5.1. Solution to the Problem

A solution to (35) is obtained using the initial conditions:

$$\text{At time t=0, v} = u\sin\alpha\,, u' = u\cos\alpha\,,\ \text{x=0, y=o} \quad (39)$$

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = -cv - g \quad (40)$$

$$\frac{dv}{v+U} = -cdt$$

$$\log\left(\frac{v+U}{u\sin\alpha+U}\right) = -ct$$

$$\frac{v+U}{u\sin\alpha+U} = e^{-ct} \quad (41)$$

$$\text{Or,} v = \frac{dy}{dt} = (u\sin\alpha + U)e^{-ct} - U \quad (42)$$

$$and\ \frac{g}{c} = U \quad (43)$$

Which is called "Terminal velocity", defined as the velocity acquired by a vertically falling body at a time when atmospheric resistance equals the gravitational force, i.e., when its velocity vanishes.

$$\frac{du\prime}{dt} = -cu' \quad (44)$$

Which, on account of the initial conditions (39), yields,

$$\log\frac{u\prime}{u\cos\alpha} = -ct$$

$$\text{Or,}u'=(ucos\alpha)e^{-ct} \tag{45}$$

Solution to (42) and (45) applying the initial conditions (39):

$$y=(u\sin\alpha + U)(1-e^{-ct})/c - Ut \tag{46}$$

$$\frac{dx}{dt} = u' = (u\cos\alpha)e^{-ct} \tag{47}$$

$$x=(u\cos\alpha)(1-e^{-ct})/c \tag{48}$$

In time T of coasting flight, the height descended $x_1 \text{Sin}\alpha$, the horizontal distance x_2' described, the horizontal velocity u_1' and vertical velocity v_1' attained are obtained by use of (48), and (46),

$$x_2'=(u\cos\alpha)(1-e^{-cT})/c \tag{49}$$

$$x_1 Sin\alpha = (u\sin\alpha + U)(1-e^{-cT})/c - UT \tag{50}$$

Attaining velocity,

$$u_1'=(u\cos\alpha)e^{-cT}$$

From (50),

$$(1-e^{-cT})/c=\frac{x_1 Sin\alpha+UT}{u\sin\alpha+U} \tag{51}$$

Which gives the time of flight in terms of the initial velocity and angle of projection. Hence, the overall horizontal range R'in this resistive medium by use of (20) and (49) becomes,

$$R'=x_1 + x_2'$$

$$=2k\frac{k(m_0-m_p)+2V_E(\sqrt{m_0}-\sqrt{m_0-m_p})}{g(\mu\cos\alpha+\sin\alpha)}\cos\alpha+(u\cos\alpha)(1-e^{-cT})/c \tag{52}$$

In order to evaluate the maximum overall range, we are to find the optimum angle α_{opt} of projection by finding,

$\frac{dR'}{d\alpha} = (a\ function\ of\ \alpha\ and \frac{dT}{d\alpha})$=0, and by use of (52), $\frac{dT}{d\alpha}$ as a function of α resulting in $\frac{dR'}{d\alpha}$ as a function of α equated to zero.

6. Projectile Motion Under Atmospheric Drag Proportional to Square of the Velocity and its Approximate Solution

The time taken by the rocket is divided into several n intervals $(t_{i-1} < t < t_i)$ corresponding to which velocity, its horizontal and vertical components gained, horizontal distances and heights attained are $(V_{i-1} < V < V_i)$ $(u_{i-1} < u' < u_i)$, $(v_{i-1} < v' < v_i)$, $(x_{i-1} < x' < x_i)$, $(y_{i-1} < y' < y_i)$. In this context Equations (37) and (38) are amended as follows:

$$\frac{du'}{dt} = \frac{d^2x'}{dt^2} = -c'Vu' \tag{53}$$

$$\frac{dv'}{dt} = \frac{d^2y'}{dt^2} = -c'Vv' - g \tag{54}$$

Which are applicable at any instant of time t in the interval (t_{i-1},t_i),c′ being the new constant of proportionality. Integrating (53) and (54) applying the construed initial conditions and initial value of V in the interval $(t_{i-1,}t_i)$ are obtained.

$$u_i = u_{i-1}e^{-c'V_{i-1}\tau} \ (t_i - t_{i-1}) = \tau \ )$$

$$v_i = v_{i-1}e^{-c'V_{i-1}\tau} \text{ (Initially neglecting g)} \tag{55}$$

$$u'_i = \frac{dx'}{dt} = u_{i-1}e^{-c'V_{i-1}t}$$

$$x'_i - x'_{i-1} = u_{i-1}\frac{1-e^{-c'V_{i-1}t}}{c'V_{i-1}}$$

$$x_i - x_{i-1} = u_{i-1}\frac{1-e^{-c'V_{i-1}\tau}}{c'V_{i-1}} \tag{56}$$

$$v'_i = \frac{dy'}{dt} = v_{i-1}e^{-c'V_{i-1}t} - gt \text{ (with approximation)}$$

$$y_i - y_{i-1} = v_{i-1}\left(\frac{1-e^{-c'V_{i-1}\tau}}{c'v_{i-1}}\right) - \frac{1}{2}g\tau^2 \tag{57}$$

Where, $$v_{i-1} = \sqrt{u_{i-1}^2 + v_{i-1}^2} \tag{58}$$

Summing up the foregoing delineations, we get the horizontal ranges and heights descended in the ith time interval

$$r_i = x_i - x_{i-1} = u_{i-1}\frac{1-e^{-c'V_{i-1}\tau}}{c'\sqrt{u_{i-1}^2+v_{i-1}^2}}$$

$$h_i = y_i - y_{i-1} = v_{i-1}(\tau - \frac{1-e^{-c'V_{i-1}\tau}}{c'\sqrt{u_{i-1}^2+v_{i-1}^2}}) - \frac{1}{2}g\tau^2$$

Hence, the total coasting range and height descended can be determined as,

$$R = \sum_1^n r_i = \sum_1^n u_{i-1}\left(\tau - \frac{1-u_{i-1}e^{-c'V_{i-1}\tau}}{c'\sqrt{u_{i-1}^2+v_{i-1}^2}}\right) \tag{59}$$

$$H = \sum_1^n h_i = \sum_1^n\{ v_{i-1}\left(\tau - \frac{1-e^{-c'V_{i-1}\tau}}{c'\sqrt{u_{i-1}^2+v_{i-1}^2}}\right) - \frac{1}{2}g\tau^2\}$$

$$= \sum_1^n\{ v_{i-1}\left(\tau - \frac{1-e^{-c'V_{i-1}\tau}}{c'\sqrt{u_{i-1}^2+v_{i-1}^2}}\right)\} - \frac{n}{2}g\tau^2\} \tag{60}$$

Where, $u_0 = u\sin\alpha$, and $v_0 = v\cos\alpha$

$$h_1 = X\sin\alpha \tag{61}$$

7. Relationships Among the Variables

Recalling Equations (29) and (30),

$$\frac{du'}{dt} = \frac{d^2x'}{dt^2} = -c'\frac{dS}{dt}u' \quad (62)$$

Then, with the initial values t=0,u′=ucosα leads to,

$$u' = u\cos\alpha e^{-C'S} \quad (63)$$

$$\frac{dv'}{dt} = \frac{d^2y'}{dt^2} = -c'\frac{dS}{dt}v' - g \quad (64)$$

Where, S is the arc distance of the projectile at any instant of time t. Integrating S as the independent variable, subject to the above initial conditions, these two equations,

$$\frac{dx}{dt} = u' = u\cos\alpha\, e^{-C'S} \quad (65)$$

$$\frac{dv'}{dS} + C'v' = -g\frac{dt}{dS} \quad (66)$$

(Integrating factor=$e^{C'S}$)

$$\text{Or, } \frac{d(v'e^{C'S})}{dS} = -ge^{C'S}\frac{1}{\sqrt{\{u\cos\alpha(e^{-C'S})\}^2 + v'^2}} \quad (67)$$

$$\text{Or, } \frac{dv'}{dS} + C'v' = -\frac{g}{\sqrt{\{u\cos\alpha(e^{-C'S})\}^2 + v'^2}} \quad (68)$$

Where,

$$\text{Velocity} = V = \frac{dS}{dt} = \sqrt{\{u\cos\alpha(e^{-C'S})\}^2 + v'^2} \quad (69)$$

Equation v'^2 is replaced in the above equation by its value calculated from (64), neglecting g because the gravitational force is small compared to the atmospheric resistance. Hence,

$$\frac{dv'}{dS} + C'v' = -\frac{g}{\sqrt{\{u\cos\alpha(e^{-C'S})\}^2 + \{u\sin\alpha(e^{-C'S})\}^2}}$$

$$\text{Or, } \frac{dv'}{dS} + C'v' = \frac{-g}{u(e^{-C'S})} \quad (70)$$

Where, with minor approximation $\frac{dS}{dt} = V = ue^{-C'S}$ (71)

Integrating factor=$e^{C'S}$ so that (70) becomes,

$$\frac{d(e^{C'S}v')}{dS} = \frac{-ge^{2C'S}}{u}$$

Integrating S from 0 to S and v′ from usinα to v′ is obtained.

$$e^{C'S}v' - u\sin\alpha = \frac{-g\{e^{2C'S}-1)}{2uC'}$$

$$\text{Or, } v' = \frac{-g\{e^{C'S}-e^{-C'S})}{2uC'} + ue^{-C'S}\sin\alpha \qquad (72)$$

Equation (72) reveals the greatest height attained when arc length S_g described is given by,

$$v'=0 \text{ ie } e^{2C'S} = 2C'\frac{u^2}{g}\sin\alpha + 1$$

$$S_g = \frac{1}{2C'}\log(2C'\frac{u^2}{g}\sin\alpha + 1) \qquad (73)$$

If h is the height at any instant of time t with the arc length S described, applying (70),

$$\text{Or, } \frac{dh}{dS} = \frac{\frac{-g\{e^{C'S}-e^{-C'S})}{2uC'}+ue^{-C'S}\sin\alpha}{\sqrt{\{u\cos\alpha(e^{-C'S})\}^2+v'^2}} \qquad (74)$$

Replacing v'^2 in the above equation, its value calculated without taking into account of small value of the gravitational force as compared to the drag, as done earlier, is acquired.

$$\frac{dh}{dS} = \frac{\frac{-g\{e^{C'S}-e^{-C'S})}{2uC'}+ue^{-C'S}\sin\alpha}{\sqrt{\{u\cos\alpha(e^{-C'S})\}^2+\{u\sin\alpha(e^{-C'S})\}^2}} = \frac{-g\{e^{2C'S}-1)}{2u^2C'} + \sin\alpha \qquad (75)$$

Integrating (75) within the limits:h from h_1 to h and S from 0 to S, the height h attained is given by,

$$h-h_1 = \frac{-g\left\{\frac{e^{2C'S}}{2C'}-S\right)}{2u^2C'} + S\sin\alpha \qquad (76)$$

From (70),

$$\frac{dS}{dt} = ue^{-C'S}$$

Carrying out the minor approximation as done in the preceding text, we get time t to cover S:

$$t = \frac{1}{uC'}(e^{C'S} - 1) \qquad (77)$$

$$\text{Or, } S = \frac{\log(utC'+1)}{C'}$$

$$\frac{dx}{dt} = u\cos\alpha e^{-C'S} \qquad (78)$$

Using (70), (78) becomes,

$$\frac{dx}{dS} = \cos\alpha \qquad (79)$$

Subject to the initial conditions (79) gives the horizontal distance covered in time t,

$$x=Sc=\frac{\log(utC'+1)}{C'}\cos\alpha \quad (80)$$

Putting h=0 and eliminating S between (76) and (80) is determined the time of flight:

$$-h_1=\frac{-g\left\{\frac{e^{2C'S}}{2C'}-S\right\}}{2u^2C'}+S\sin\alpha$$

$$\text{Or, } \frac{-g\left\{\frac{e^{2\log(utC'+1)}}{2C'}-\frac{\log(utC'+1)}{C'}\right\}}{2u^2C'}+\frac{\log(utC'+1)}{C'}\sin\alpha+h_1=0$$

$$\text{Or, } \frac{-g\left\{\frac{(utC'+1)^2}{2C'}-\frac{\log(utC'+1)}{C'}\right\}}{2u^2C'}+\frac{\log(utC'+1)}{C'}\sin\alpha+h_1=0 \quad (81)$$

The value of t from (81) gives the time of flight, which, on substitution into (80), gives the horizontal range. Equation (70) can be modified with a more appropriate inclusion of gravitational force by use of v′ expressed by (72):

$$V^2=\left(\frac{dS}{dt}\right)^2=u'^2+v'^2=(u\cos\alpha e^{-C'S})^2+\left\{\frac{-g\left\{(e^{C'S}-e^{-C'S}\right)}{2uC'}+ue^{-C'S}\sin\alpha\right\}^2$$

$$=(ue^{-C'S})^2+\left\{\frac{g\left\{(e^{C'S}-e^{-C'S}\right)}{2uC'}\right\}^2-\frac{g\left\{(1-e^{-2C'S}\right)}{C'}\sin\alpha$$

$$=e^{-2C'S}u^2+\frac{g^2(e^{2C'S}+e^{-2C'S}-2)}{4u^2C'^2}-\frac{g\left\{(1-e^{-2C'S}\right)}{C'}\sin\alpha \quad (82)$$

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Appendix

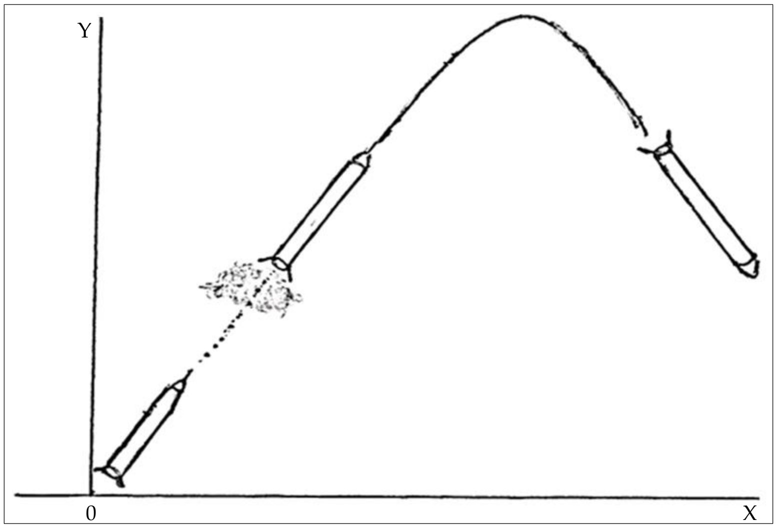

Fig. 1 Rocket launched followed by coasting flight

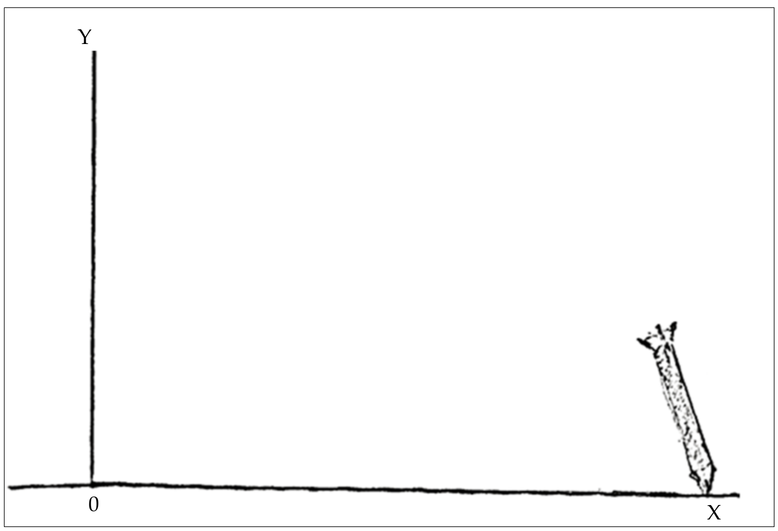

Fig. 2 Rocket hitting the ground after coasting flight