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*Original Article*

# An Equally Spaced Numerical Method for Solving Volterra Integro-Differential Equations of Second Kind

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**Abstract -** The researchers created an Equally Spaced New Numerical Method (NNM), which functions as a highly effective tool for solving Volterra integro-differential equations of the second kind that scientists use to model physical, engineering, and biological systems with memory effects. The new method derives from using a linear block algorithm, which implements the third derivative together with a linear multistep framework to construct a hybrid block scheme that operates on equally spaced nodes. The formulation delivers two numerical schemes that operate continuously and discretely, and their coefficients are derivable through analytical methods. The method undergoes a comprehensive theoretical examination, which establishes its fundamental characteristics, including order and error constant, together with consistency, zero-stability, and convergence and region of absolute stability. The method evaluation uses numerical tests on standard problems, which demonstrate that the NNM produces results that match analytical solutions while achieving lower error rates than existing numerical methods, which use trapezoidal-type and block methods. The NNM functions as a dependable and competitive numerical method for Volterra integro-differential equations of the second kind, according to the results, which support its accuracy and stability and robustness demonstrated through tabulated data and graphical representations.

**Keywords -** Equally Spaced Numerical Method, Volterra Integro-Differential Equations, Linear Block Algorithm, Numerical Stability, Convergence Analysis.

## 1. Introduction

Scientists and engineers use mathematical models to create differential and integral equations that represent physical and natural processes. The system behaviors of different phenomena are modeled through differential equations, which include stiff, oscillatory and fractional types, while integral equations describe system behavior based on its past states. The integral equations are divided into two categories, Fredholm and Volterra, based on their integration limit characteristics, which are essential to the study of fluid dynamics, heat transfer, and electromagnetic fields [1]. The development of multiple analytical and numerical solution methods, which include the Adomian Decomposition Method and collocation techniques, emerged as a result of integral equations that require both accurate and effective solutions [1, 2]. Volterra Integro-Differential Equations (VIDEs) represent mathematical models that demonstrate systems that determine their future states by using current change rates and total past system behavior. The structure of VIDEs unites differential and integral equations because they require unknown functions to exist both in derivatives and in integral signs, making them suitable for modeling processes that depend on past memory. Their applications include population dynamics, viscoelasticity, and heat conduction. The equations of this system treat previous elements as determining factors that shape the system's development, while they function as alternative differential equations because they include integral components. The following is a standard representation of an equation in such a form:

$$\rho(\xi) = \vartheta(\xi) + \varphi \int_{\omega(\xi)}^{\varpi(\xi)} \mathrm{K}(\xi, \tau)\rho(\tau)d\tau \tag{1}$$

Where $\varphi$ is a constant parameter, $K(\xi,\ \tau)$ is called the kernel of the integral equation, $\vartheta(\xi)$ is a function, and $\omega(\xi)$ and $\varpi(\xi)$ are the limits of integration, which can be constants, variables, or a combination of both [1]. This work focuses on a class of integro-differential equations of Volterra type, which are characterized by their hereditary structure and are expressed in the form presented in the referenced equation (2) as:

$$\rho^{(u)}(\xi) = \vartheta(\xi) + \phi \int_{\kappa}^{\xi} K(\xi,\ \tau)\rho(\tau)\, d\tau \tag{2}$$

Classical techniques for solving Volterra Integro-Differential Equations (VIDEs) include series expansions, successive approximations like Picard iteration, and the Laplace transform. The basic techniques have served as the primary method for scientists to analyze linear VIDEs, which exist in solution spaces described in [1, 5]. The method faces major challenges because it requires substantial computational power and struggles to achieve advanced accuracy, and the series results do not accurately represent the actual physical processes that scientists aim to study [1, 6, 7]. The Adomian Decomposition Method (ADM), Homotopy Analysis Method (HAM), and Variational Iteration Method (VIM) developed better semi-analytical techniques that function as dependable instruments for solving nonlinear VIDEs because these methods provide researchers with increased solution options and better accuracy. The field has adopted three different approaches, which include perturbation methods and Green's function and resolvent kernel techniques to address particular problem types. The development of precise yet powerful numerical and hybrid methods has increased in importance because existing advanced techniques struggle to solve complex systems that exhibit stiffness characteristics [11, 12]. Researchers have historically used classical techniques, which include series solutions, successive approximations, and Laplace transforms to solve VIDs [1]. The core problem of these methods remains because they require extensive computations and advanced mathematical techniques to produce results that do not match real-world conditions. The development of semi-analytical methods has advanced through the introduction of the Adaptive Domain Method [4, 8], the Homotopy Analysis Method [13, 14], the Homotopy Perturbation Method [15], and the Variation Iteration Method [16]. Nonlinear systems are the primary beneficiaries of these methods as they provide more reliable and flexible solutions. Yet, the problem with series convergence still remains, as certain problems converge very quickly, while others have to go through many more terms, thereby increasing the effort needed from humans and the load on computers [17].

Definition 1: The general form of a linear multistep method is given as:

$$\sum_{j=0}^{k} \alpha_j\, y_{n+j} = h^d \sum_{j=0}^{k} \beta_j\, f_{n+j} \tag{3}$$

where

$$f_{n+j} = f\left(x_{n+j}, y_{n+j}, \cdots, y_{n+j}^{d-1}\right)$$

$$y_{n+j} = y\left(x_{n+j}\right), j = 0(1)k$$

$d$ is the order of the differential equation, $\alpha_j$ and $\beta_j$ are real constants where both $\alpha_0$and $\beta_0$ are not zero, equation (1.3) can be represented by:

$$\rho(r) = \sum_{j=0}^{k} \alpha_j\, y_{n+j} \text{ and } \rho(r) = h^d \sum_{j=0}^{k} \beta_j\, f_{n+j}$$

$\rho(r)$ and $\sigma(r)$ are known as the first and second characteristic polynomials, respectively.

Equation (2) is said to be implicit if $\beta_j \neq 0$ that is, the approximate solution at $x_{n+k}$, which is $y_{n+k}$ reflected on both sides of (2). On the other hand, (1.2) is explicit if$\beta_j = 0$, that is, the approximate value of $y_{n+j}$ can directly be determined in terms of $y_{n+j}, f_{n+j}, j = 0(1)k-1$ the implicit method, which requires the determination of initial values for $y_{n+j}, y'_{n+k}, \cdots, y_{n+k}^{d-1}$ in terms of$f\left(x_{n+k}, y_{n+k}, \cdots, y_{n+k}^{d-1}\right)$.

## 2. Derivation of New Numerical Method (NNM)

The New Numerical Method (NNM) was a product of the application of the third derivative linear block algorithm for the numerical resolution of the Volterra integro-differential equation of the second kind. To arrive at the NNM, we treated proposition 2.1 as a Linear Block Algorithm (LBA) using the techniques of [18]. We consider the general linear multistep method of the type:

$$\sum_{j=0}^{1} \alpha_j \rho_{n+j} = h^{\mu} \sum_{j=0}^{1} \beta_j \vartheta_{n+j} \tag{4}$$

### 2.1. Proposition 1

Think about the universal linear multistep method (4) along with the single step block hybrid method, where a numerical scheme exists with the linear block algorithm having the shape of:

$$\rho_{n+\eta} = \sum_{j=0}^{2} \frac{(\eta h)^j}{j!} \rho_n^{(j)} + \sum_{j=0}^{1}\left(\Lambda_{j\eta} \vartheta_{n+j}\right), \quad \eta = -\tfrac{1}{7},\ -\tfrac{2}{7},\ 0,\ \tfrac{1}{7},\ \tfrac{2}{7},\ \tfrac{3}{7},\ \tfrac{4}{7},\ \tfrac{5}{7},\ \tfrac{6}{7},\ 1 \tag{5}$$

together with its higher derivatives

$$\rho_{n+\eta}^{\sigma} = \sum_{j=0}^{2-\tau} \frac{(\eta h)^j}{j!} \rho_n^{(j+\sigma)} + \sum_{j=0}^{7}\left(X_{j\sigma} \vartheta_{n+j}\right),\ \sigma = 1_{\left(-\frac{1}{7}, -\frac{2}{7}, 0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, 1\right)},\ \sigma = 2_{\left(-\frac{1}{7}, -\frac{2}{7}, 0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, 1\right)} \tag{6}$$

is considered, with $\Lambda_{\eta j} = \Psi^{-1} Z$ and $X_{\eta j \sigma} = \Psi^{-1} E$ where

$$\Psi = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\frac{\left(-\frac{1}{7}h\right)^1}{1!} & \frac{\left(-\frac{2}{7}h\right)^1}{1!} & \frac{(0)^1}{1!} & \frac{\left(\frac{1}{7}h\right)^1}{1!} & \frac{\left(\frac{2}{7}h\right)^1}{1!} & \frac{\left(\frac{3}{7}h\right)^1}{1!} & \frac{\left(\frac{4}{7}h\right)^1}{1!} & \frac{\left(\frac{5}{7}h\right)^1}{1!} & \frac{\left(\frac{6}{7}h\right)^1}{1!} & \frac{(h)^1}{1!} \\
\frac{\left(-\frac{1}{7}h\right)^2}{2!} & \frac{\left(-\frac{2}{7}h\right)^2}{2!} & \frac{(0)^2}{2!} & \frac{\left(\frac{1}{7}h\right)^2}{2!} & \frac{\left(\frac{2}{7}h\right)^2}{2!} & \frac{\left(\frac{3}{7}h\right)^2}{2!} & \frac{\left(\frac{4}{7}h\right)^2}{2!} & \frac{\left(\frac{5}{7}h\right)^2}{2!} & \frac{\left(\frac{6}{7}h\right)^2}{2!} & \frac{(h)^2}{2!} \\
\frac{\left(-\frac{1}{7}h\right)^3}{3!} & \frac{\left(-\frac{2}{7}h\right)^3}{3!} & \frac{(0)^3}{3!} & \frac{\left(\frac{1}{7}h\right)^3}{3!} & \frac{\left(\frac{2}{7}h\right)^3}{3!} & \frac{\left(\frac{3}{7}h\right)^3}{3!} & \frac{\left(\frac{4}{7}h\right)^3}{3!} & \frac{\left(\frac{5}{7}h\right)^3}{3!} & \frac{\left(\frac{6}{7}h\right)^3}{3!} & \frac{(h)^3}{3!} \\
\frac{\left(-\frac{1}{7}h\right)^4}{4!} & \frac{\left(-\frac{2}{7}h\right)^4}{4!} & \frac{(0)^4}{4!} & \frac{\left(\frac{1}{7}h\right)^4}{4!} & \frac{\left(\frac{2}{7}h\right)^4}{4!} & \frac{\left(\frac{3}{7}h\right)^4}{4!} & \frac{\left(\frac{4}{7}h\right)^4}{4!} & \frac{\left(\frac{5}{7}h\right)^4}{4!} & \frac{\left(\frac{6}{7}h\right)^4}{4!} & \frac{(h)^4}{4!} \\
\frac{\left(-\frac{1}{7}h\right)^5}{5!} & \frac{\left(-\frac{2}{7}h\right)^5}{5!} & \frac{(0)^5}{5!} & \frac{\left(\frac{1}{7}h\right)^5}{5!} & \frac{\left(\frac{2}{7}h\right)^5}{5!} & \frac{\left(\frac{3}{7}h\right)^5}{5!} & \frac{\left(\frac{4}{7}h\right)^5}{5!} & \frac{\left(\frac{5}{7}h\right)^5}{5!} & \frac{\left(\frac{6}{7}h\right)^5}{5!} & \frac{(h)^5}{5!} \\
\frac{\left(-\frac{1}{7}h\right)^6}{6!} & \frac{\left(-\frac{2}{7}h\right)^6}{6!} & \frac{(0)^6}{6!} & \frac{\left(\frac{1}{7}h\right)^6}{6!} & \frac{\left(\frac{2}{7}h\right)^6}{6!} & \frac{\left(\frac{3}{7}h\right)^6}{6!} & \frac{\left(\frac{4}{7}h\right)^6}{6!} & \frac{\left(\frac{5}{7}h\right)^6}{6!} & \frac{\left(\frac{6}{7}h\right)^6}{6!} & \frac{(h)^6}{6!} \\
\frac{\left(-\frac{1}{7}h\right)^7}{7!} & \frac{\left(-\frac{2}{7}h\right)^7}{7!} & \frac{(0)^7}{7!} & \frac{\left(\frac{1}{7}h\right)^7}{7!} & \frac{\left(\frac{2}{7}h\right)^7}{7!} & \frac{\left(\frac{3}{7}h\right)^7}{7!} & \frac{\left(\frac{4}{7}h\right)^7}{7!} & \frac{\left(\frac{5}{7}h\right)^7}{7!} & \frac{\left(\frac{6}{7}h\right)^7}{7!} & \frac{(h)^7}{7!} \\
\frac{\left(-\frac{1}{7}h\right)^8}{8!} & \frac{\left(-\frac{2}{7}h\right)^8}{8!} & \frac{(0)^8}{8!} & \frac{\left(\frac{1}{7}h\right)^8}{8!} & \frac{\left(\frac{2}{7}h\right)^8}{8!} & \frac{\left(\frac{3}{7}h\right)^8}{8!} & \frac{\left(\frac{4}{7}h\right)^8}{8!} & \frac{\left(\frac{5}{7}h\right)^8}{8!} & \frac{\left(\frac{6}{7}h\right)^8}{8!} & \frac{(h)^8}{8!} \\
\frac{\left(-\frac{1}{7}h\right)^9}{9!} & \frac{\left(-\frac{2}{7}h\right)^9}{9!} & \frac{(0)^9}{9!} & \frac{\left(\frac{1}{7}h\right)^9}{9!} & \frac{\left(\frac{2}{7}h\right)^9}{9!} & \frac{\left(\frac{3}{7}h\right)^9}{9!} & \frac{\left(\frac{4}{7}h\right)^9}{9!} & \frac{\left(\frac{5}{7}h\right)^9}{9!} & \frac{\left(\frac{6}{7}h\right)^9}{9!} & \frac{(h)^9}{9!}
\end{pmatrix},\ Z = \begin{pmatrix}
\frac{(\eta h)^3}{3!} \\ \frac{(\eta h)^4}{4!} \\ \frac{(\eta h)^5}{5!} \\ \frac{(\eta h)^6}{6!} \\ \frac{(\eta h)^7}{7!} \\ \frac{(\eta h)^8}{8!} \\ \frac{(\eta h)^9}{9!} \\ \frac{(\eta h)^{10}}{10!} \\ \frac{(\eta h)^{11}}{11!} \\ \frac{(\eta h)^{12}}{12!}
\end{pmatrix},\ E = \begin{pmatrix}
\frac{(\eta h)^{3-\sigma}}{(3-\sigma)!} \\ \frac{(\eta h)^{4-\sigma}}{(4-\sigma)!} \\ \frac{(\eta h)^{5-\sigma}}{(5-\sigma)!} \\ \frac{(\eta h)^{6-\sigma}}{(6-\sigma)!} \\ \frac{(\eta h)^{7-\sigma}}{(7-\sigma)!} \\ \frac{(\eta h)^{8-\sigma}}{(8-\sigma)!} \\ \frac{(\eta h)^{9-\sigma}}{(9-\sigma)!} \\ \frac{(\eta h)^{10-\sigma}}{(10-\sigma)!} \\ \frac{(\eta h)^{11-\sigma}}{(11-\sigma)!} \\ \frac{(\eta h)^{12-\sigma}}{(12-\sigma)!}
\end{pmatrix}$$

Proof: Solving equations (5) and (6) one by one to yield the continuous schemes in the form of a polynomial as:

$$\rho(\xi_n+\eta h)=\alpha_{\frac{1}{7}}\rho_{n+\frac{1}{7}}+\alpha_{\frac{2}{7}}\rho_{n+\frac{2}{7}}+\alpha_{\frac{3}{7}}\rho_{n+\frac{3}{7}}+ h^3\left(\beta_{-\frac{1}{7}}\vartheta_{n-\frac{1}{7}}+\beta_{-\frac{2}{7}}\vartheta_{n-\frac{2}{7}}+\beta_0\vartheta_n+\beta_{\frac{1}{7}}\vartheta_{n+\frac{1}{7}}+\beta_{\frac{2}{7}}\vartheta_{n+\frac{2}{7}}+\beta_{\frac{3}{7}}\vartheta_{n+\frac{3}{7}}+\beta_{\frac{4}{7}}\vartheta_{n+\frac{4}{7}}+\beta_{\frac{5}{7}}\vartheta_{n+\frac{5}{7}}+\beta_{\frac{6}{7}}\vartheta_{n+\frac{6}{7}}+\beta_1\vartheta_{n+1}\right) \quad (7)$$

Where $\eta=\xi_n+\xi h$ in the equation (7) and,

$$\left.\begin{aligned}
&\alpha_{\frac{1}{7}}=3-\frac{35}{2}\eta+\frac{49}{2}\eta^2,\ \alpha_{\frac{2}{7}}=-3+28\eta-49\eta^2,\ \alpha_{\frac{1}{7}}=1-\frac{21}{2}\eta+\frac{49}{2}\eta^2,\\
&\beta_{-\frac{1}{7}}=-\frac{803}{165957120}-\frac{30571}{558835200}\eta+\frac{2734121}{2235340800}\eta^2-\frac{7}{96}\eta^4+\frac{2051}{4800}\eta^5-\frac{161651}{172800}\eta^6+\frac{9947}{86400}\eta^7-\frac{977207}{276480}\eta^8+\frac{789929}{103680}\eta^9\\
&-\frac{15647317}{2073600}\eta^{10}+\frac{10706059}{2851200}\eta^{11}-\frac{5764801}{7603200}\eta^{12}\\
&\beta_{-\frac{1}{3}}=\frac{373}{829785600}+\frac{37}{17463600}\eta-\frac{499519}{6706022400}\eta^2+\frac{7}{1728}\eta^4-\frac{1561}{86400}\eta^5+\frac{49}{97200}\eta^6-\frac{34643}{2800}\eta^7+\frac{549829}{829440}\eta^8-\frac{218491}{207360}\eta^9\\
&+\frac{5764801}{6220800}\eta^{10}-\frac{823543}{1900800}\eta^{11}+\frac{5764801}{68428800}\eta^{12}\\
&\beta_0=-\frac{11}{29635200}+\frac{160309}{97796160}\eta-\frac{1048601}{37255680}\eta^2+\frac{1}{6}\eta^3-\frac{15}{160}\eta^4+\frac{553}{864}\eta^5-\frac{62671}{17280}\eta^6+\frac{10633}{2880}\eta^7-\frac{175273}{23040}\eta^8+\frac{84035}{3456}\eta^9\\
&-\frac{117649}{4320}\eta^{10}+\frac{823543}{57024}\eta^{11}-\frac{5764801}{1900800}\eta^{12}\\
&\beta_{\frac{1}{7}}=-\frac{292309}{207446400}+\frac{5039777}{244490400}\eta-\frac{48525139}{558835200}\eta^2+\frac{49}{72}\eta^4-\frac{637}{3600}\eta^5-\frac{757687}{129600}\eta^6+\frac{226037}{21600}\eta^7-\frac{559433}{69120}\eta^8+\frac{2336173}{151840\ 11}\eta^9\\
&-\frac{823543}{14400}\eta^{10}+\frac{5764801}{178200}\eta^{11}-\frac{40353607}{5702400}\eta^{12}\\
&\beta_{\frac{2}{7}}=-\frac{651377}{414892800}+\frac{539591}{34927200}\eta-\frac{27575231}{1117670400}\eta^2-\frac{49}{96}\eta^4+\frac{4067}{4800}\eta^5+\frac{466823}{86400}\eta^6-\frac{1259839}{86400}\eta^7+\frac{501809}{138240}\eta^8-\frac{5563117}{103680}\eta^9\\
&+\frac{79883671}{1036800}\eta^{10}-\frac{132590423}{2851200}\eta^{11}+\frac{40353607}{3801600}\eta^{12}\\
&\beta_{\frac{3}{7}}=\frac{47531}{414892800}-\frac{80039}{177811200}\eta-\frac{116047}{17740800}\eta^2+\frac{49}{144}\eta^4-\frac{15631}{21600}\eta^5+\frac{298753}{86400}\eta^6-\frac{177331}{14400}\eta^7+\frac{40817}{46080}\eta^8-\frac{184877}{4320}\eta^9\\
&+\frac{23882747}{345600}\eta^{10}-\frac{5764801}{129600}\eta^{11}+\frac{40353607}{3801600}\eta^{12}\\
&\beta_{\frac{4}{7}}=-\frac{517}{8297856}+\frac{138529}{488980800}\eta+\frac{567667}{186278400}\eta^2-\frac{49}{288}\eta^4+\frac{2891}{7200}\eta^5+\frac{429779}{259200}\eta^6-\frac{293951}{43200}\eta^7+\frac{45619}{23040}\eta^8+\frac{1193297}{51840}\eta^9\\
&-\frac{10706059}{259200}\eta^{10}+\frac{40353607}{1425600}\eta^{11}-\frac{40353607}{5702400}\eta^{12}\\
&\beta_{\frac{5}{7}}=\frac{83}{4233600}-\frac{1123}{13970880}\eta-\frac{118931}{111767040}\eta^2+\frac{7}{120}\eta^4-\frac{7}{48}\eta^5-\frac{4753}{8640}\eta^6+\frac{10633}{4320}\eta^7-\frac{74431}{69120}\eta^8-\frac{84035}{10368}\eta^9+\frac{823543}{51840}\eta^{10}\\
&-\frac{823543}{71280}\eta^{11}+\frac{5764801}{1900800}\eta^{12}\\
&\beta_{\frac{6}{7}}=-\frac{3179}{829785600}+\frac{877}{61122600}\eta+\frac{167257}{745113600}\eta^2-\frac{7}{576}\eta^4+\frac{2723}{86400}\eta^5+\frac{9653}{86400}\eta^6-\frac{30527}{57600}\eta^7+\frac{26411}{92160}\eta^8+\frac{117649}{69120}\eta^9-\\
&\frac{823543}{230400}\eta^{10}+\frac{15647317}{5702400}\eta^{11}-\frac{5764801}{7603200}\eta^{12}\\
&\beta_1=\frac{289}{829785600}-\frac{1579}{1303948800}\eta-\frac{144509}{6706022400}\eta^2+\frac{1}{864}\eta^4-\frac{133}{43200}\eta^5-\frac{16219}{1555200}\eta^6+\frac{4459}{86400}\eta^7-\frac{26411}{829440}\eta^8-\frac{16807}{103680}\eta^9\\
&+\frac{2235331}{6220800}\eta^{10}-\frac{823543}{2851200}\eta^{11}+\frac{5764801}{68428800}\eta^{12}
\end{aligned}\right\} \quad (8)$$

Expanding the generalized algorithm (5) to give an NNM as:

$$\left.\begin{aligned}
\rho_{n-\frac{1}{7}} &= \rho_n - \frac{1}{7}h\rho'_n + \frac{\left(-\frac{1}{7}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{10}\vartheta_{n-\frac{1}{7}} + \Lambda_{11}\vartheta_{n-\frac{2}{7}} + \Lambda_{12}\vartheta_n + \Lambda_{13}\vartheta_{n+\frac{1}{7}} + \Lambda_{14}\vartheta_{n+\frac{2}{7}} + \Lambda_{15}\vartheta_{n+\frac{3}{7}} + \Lambda_{16}\vartheta_{n+\frac{4}{7}} + \Lambda_{17}\vartheta_{n+\frac{5}{7}} + \Lambda_{18}\vartheta_{n+\frac{6}{7}} + \Lambda_{19}\vartheta_{n+1}\right)\\
\rho_{n-\frac{2}{7}} &= \rho_n - \frac{2}{7}h\rho'_n + \frac{\left(-\frac{2}{7}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{20}\vartheta_{n-\frac{1}{7}} + \Lambda_{21}\vartheta_{n-\frac{2}{7}} + \Lambda_{22}\vartheta_n + \Lambda_{23}\vartheta_{n+\frac{1}{7}} + \Lambda_{24}\vartheta_{n+\frac{2}{7}} + \Lambda_{25}\vartheta_{n+\frac{3}{7}} + \Lambda_{26}\vartheta_{n+\frac{4}{7}} + \Lambda_{27}\vartheta_{n+\frac{5}{7}} + \Lambda_{28}\vartheta_{n+\frac{6}{7}} + \Lambda_{29}\vartheta_{n+1}\right)\\
\rho_{n+\frac{1}{7}} &= \rho_n + \frac{1}{7}h\rho'_n + \frac{\left(\frac{1}{7}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{30}\vartheta_{n-\frac{1}{7}} + \Lambda_{31}\vartheta_{n-\frac{2}{7}} + \Lambda_{32}\vartheta_n + \Lambda_{33}\vartheta_{n+\frac{1}{7}} + \Lambda_{34}\vartheta_{n+\frac{2}{7}} + \Lambda_{35}\vartheta_{n+\frac{3}{7}} + \Lambda_{36}\vartheta_{n+\frac{4}{7}} + \Lambda_{37}\vartheta_{n+\frac{5}{7}} + \Lambda_{38}\vartheta_{n+\frac{6}{7}} + \Lambda_{39}\vartheta_{n+1}\right)\\
\rho_{n+\frac{2}{7}} &= \rho_n + \frac{2}{7}h\rho'_n + \frac{\left(\frac{2}{7}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{40}\vartheta_{n-\frac{1}{7}} + \Lambda_{41}\vartheta_{n-\frac{2}{7}} + \Lambda_{42}\vartheta_n + \Lambda_{43}\vartheta_{n+\frac{1}{7}} + \Lambda_{44}\vartheta_{n+\frac{2}{7}} + \Lambda_{45}\vartheta_{n+\frac{3}{7}} + \Lambda_{46}\vartheta_{n+\frac{4}{7}} + \Lambda_{47}\vartheta_{n+\frac{5}{7}} + \Lambda_{48}\vartheta_{n+\frac{6}{7}} + \Lambda_{49}\vartheta_{n+1}\right)\\
\rho_{n+\frac{3}{7}} &= \rho_n + \frac{3}{7}h\rho'_n + \frac{\left(\frac{3}{7}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{50}\vartheta_{n-\frac{1}{7}} + \Lambda_{51}\vartheta_{n-\frac{2}{7}} + \Lambda_{52}\vartheta_n + \Lambda_{53}\vartheta_{n+\frac{1}{7}} + \Lambda_{54}\vartheta_{n+\frac{2}{7}} + \Lambda_{55}\vartheta_{n+\frac{3}{7}} + \Lambda_{56}\vartheta_{n+\frac{4}{7}} + \Lambda_{57}\vartheta_{n+\frac{5}{7}} + \Lambda_{58}\vartheta_{n+\frac{6}{7}} + \Lambda_{59}\vartheta_{n+1}\right)\\
\rho_{n+\frac{4}{7}} &= \rho_n + \frac{4}{7}h\rho'_n + \frac{\left(\frac{4}{7}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{60}\vartheta_{n-\frac{1}{7}} + \Lambda_{61}\vartheta_{n-\frac{2}{7}} + \Lambda_{62}\vartheta_n + \Lambda_{63}\vartheta_{n+\frac{1}{7}} + \Lambda_{64}\vartheta_{n+\frac{2}{7}} + \Lambda_{65}\vartheta_{n+\frac{3}{7}} + \Lambda_{66}\vartheta_{n+\frac{4}{7}} + \Lambda_{67}\vartheta_{n+\frac{5}{7}} + \Lambda_{68}\vartheta_{n+\frac{6}{7}} + \Lambda_{69}\vartheta_{n+1}\right)\\
\rho_{n+\frac{5}{7}} &= \rho_n + \frac{5}{7}h\rho'_n + \frac{\left(\frac{5}{7}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{70}\vartheta_{n-\frac{1}{7}} + \Lambda_{71}\vartheta_{n-\frac{2}{7}} + \Lambda_{72}\vartheta_n + \Lambda_{73}\vartheta_{n+\frac{1}{7}} + \Lambda_{74}\vartheta_{n+\frac{2}{7}} + \Lambda_{75}\vartheta_{n+\frac{3}{7}} + \Lambda_{76}\vartheta_{n+\frac{4}{7}} + \Lambda_{77}\vartheta_{n+\frac{5}{7}} + \Lambda_{78}\vartheta_{n+\frac{6}{7}} + \Lambda_{79}\vartheta_{n+1}\right)\\
\rho_{n+\frac{6}{7}} &= \rho_n + \frac{6}{7}h\rho'_n + \frac{\left(\frac{6}{7}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{80}\vartheta_{n-\frac{1}{7}} + \Lambda_{81}\vartheta_{n-\frac{2}{7}} + \Lambda_{82}\vartheta_n + \Lambda_{83}\vartheta_{n+\frac{1}{7}} + \Lambda_{84}\vartheta_{n+\frac{2}{7}} + \Lambda_{85}\vartheta_{n+\frac{3}{7}} + \Lambda_{86}\vartheta_{n+\frac{4}{7}} + \Lambda_{87}\vartheta_{n+\frac{5}{7}} + \Lambda_{88}\vartheta_{n+\frac{6}{7}} + \Lambda_{89}\vartheta_{n+1}\right)\\
\rho_{n+1} &= \rho_n + h\rho'_n + \frac{(h)^2}{2!}\rho''_n + h^3\left(\Lambda_{90}\vartheta_{n-\frac{1}{7}} + \Lambda_{91}\vartheta_{n-\frac{2}{7}} + \Lambda_{92}\vartheta_n + \Lambda_{93}\vartheta_{n+\frac{1}{7}} + \Lambda_{94}\vartheta_{n+\frac{2}{7}} + \Lambda_{95}\vartheta_{n+\frac{3}{7}} + \Lambda_{96}\vartheta_{n+\frac{4}{7}} + \Lambda_{97}\vartheta_{n+\frac{5}{7}} + \Lambda_{98}\vartheta_{n+\frac{6}{7}} + \Lambda_{99}\vartheta_{n+1}\right)
\end{aligned}\right\} \tag{9}$$

Also, expand the higher derivative of the generalized (9)

$$\left.\begin{aligned}
\rho'_{n-\frac{1}{7}} &= \rho'_n - \frac{1}{7}h\rho''_n + h^2 f\left(X_{110}\vartheta_{n-\frac{1}{7}} + X_{111}\vartheta_{n-\frac{2}{7}} + X_{112}\vartheta_n + X_{113}\vartheta_{n+\frac{1}{7}} + X_{114}\vartheta_{n+\frac{2}{7}} + X_{115}\vartheta_{n+\frac{3}{7}} + X_{116}\vartheta_{n+\frac{4}{7}} + X_{117}\vartheta_{n+\frac{5}{7}} + X_{118}\vartheta_{n+\frac{6}{7}} + X_{119}\vartheta_{n+1}\right)\\
\rho'_{n-\frac{2}{7}} &= \rho'_n - \frac{2}{7}h\rho''_n + h^2\left(X_{120}\vartheta_{n-\frac{1}{7}} + X_{121}\vartheta_{n-\frac{2}{7}} + X_{122}\vartheta_n + X_{123}\vartheta_{n+\frac{1}{7}} + X_{124}\vartheta_{n+\frac{2}{7}} + X_{125}\vartheta_{n+\frac{3}{7}} + X_{126}\vartheta_{n+\frac{4}{7}} + X_{127}\vartheta_{n+\frac{5}{7}} + X_{128}\vartheta_{n+\frac{6}{7}} + X_{129}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{1}{7}} &= \rho'_n + \frac{1}{7}h\rho''_n + h^2\left(X_{130}\vartheta_{n-\frac{1}{7}} + X_{131}\vartheta_{n-\frac{2}{7}} + X_{132}\vartheta_n + X_{133}\vartheta_{n+\frac{1}{7}} + X_{134}\vartheta_{n+\frac{2}{7}} + X_{135}\vartheta_{n+\frac{3}{7}} + X_{136}\vartheta_{n+\frac{4}{7}} + X_{137}\vartheta_{n+\frac{5}{7}} + X_{138}\vartheta_{n+\frac{6}{7}} + X_{139}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{2}{7}} &= \rho'_n + \frac{2}{7}h\rho''_n + h^2\left(X_{140}\vartheta_{n-\frac{1}{7}} + X_{141}\vartheta_{n-\frac{2}{7}} + X_{142}\vartheta_n + X_{143}\vartheta_{n+\frac{1}{7}} + X_{144}\vartheta_{n+\frac{2}{7}} + X_{145}\vartheta_{n+\frac{3}{7}} + X_{146}\vartheta_{n+\frac{4}{7}} + X_{147}\vartheta_{n+\frac{5}{7}} + X_{148}\vartheta_{n+\frac{6}{7}} + X_{149}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{3}{7}} &= \rho'_n + \frac{3}{7}h\rho''_n + h^2\left(X_{150}\vartheta_{n-\frac{1}{7}} + X_{151}\vartheta_{n-\frac{2}{7}} + X_{152}\vartheta_n + X_{153}\vartheta_{n+\frac{1}{7}} + X_{154}\vartheta_{n+\frac{2}{7}} + X_{155}\vartheta_{n+\frac{3}{7}} + X_{156}\vartheta_{n+\frac{4}{7}} + X_{157}\vartheta_{n+\frac{5}{7}} + X_{158}\vartheta_{n+\frac{6}{7}} + X_{159}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{4}{7}} &= \rho'_n + \frac{4}{7}h\rho''_n + h^2\left(X_{160}\vartheta_{n-\frac{1}{7}} + X_{161}\vartheta_{n-\frac{2}{7}} + X_{162}\vartheta_n + X_{163}\vartheta_{n+\frac{1}{7}} + X_{164}\vartheta_{n+\frac{2}{7}} + X_{165}\vartheta_{n+\frac{3}{7}} + X_{166}\vartheta_{n+\frac{4}{7}} + X_{167}\vartheta_{n+\frac{5}{7}} + X_{168}\vartheta_{n+\frac{6}{7}} + X_{169}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{5}{7}} &= \rho'_n + \frac{5}{7}h\rho''_n + h^2\left(X_{170}\vartheta_{n-\frac{1}{7}} + X_{171}\vartheta_{n-\frac{2}{7}} + X_{172}\vartheta_n + X_{173}\vartheta_{n+\frac{1}{7}} + X_{174}\vartheta_{n+\frac{2}{7}} + X_{175}\vartheta_{n+\frac{3}{7}} + X_{176}\vartheta_{n+\frac{4}{7}} + X_{177}\vartheta_{n+\frac{5}{7}} + X_{178}\vartheta_{n+\frac{6}{7}} + X_{179}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{6}{7}} &= \rho'_n + \frac{5}{7}h\rho''_n + h^2\left(X_{180}\vartheta_{n-\frac{1}{7}} + X_{181}\vartheta_{n-\frac{2}{7}} + X_{182}\vartheta_n + X_{183}\vartheta_{n+\frac{1}{7}} + X_{184}\vartheta_{n+\frac{2}{7}} + X_{185}\vartheta_{n+\frac{3}{7}} + X_{186}\vartheta_{n+\frac{4}{7}} + X_{187}\vartheta_{n+\frac{5}{7}} + X_{188}\vartheta_{n+\frac{6}{7}} + X_{189}\vartheta_{n+1}\right)\\
\rho'_{n+1} &= \rho'_n + h\rho''_n + h^2\left(X_{190}\vartheta_{n-\frac{1}{7}} + X_{191}\vartheta_{n-\frac{2}{7}} + X_{192}\vartheta_n + X_{193}\vartheta_{n+\frac{1}{7}} + X_{194}\vartheta_{n+\frac{2}{7}} + X_{195}\vartheta_{n+\frac{3}{7}} + X_{196}\vartheta_{n+\frac{4}{7}} + X_{197}\vartheta_{n+\frac{5}{7}} + X_{198}\vartheta_{n+\frac{6}{7}} + X_{199}\vartheta_{n+1}\right)
\end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned}
\rho''_{n-\frac{1}{7}} &= \rho''_n + h\left(X_{210}\vartheta_{n-\frac{1}{7}} + X_{211}\vartheta_{n-\frac{2}{7}} + X_{212}\vartheta_n + X_{213}\vartheta_{n+\frac{1}{7}} + X_{214}\vartheta_{n+\frac{2}{7}} + X_{215}\vartheta_{n+\frac{3}{7}} + X_{216}\vartheta_{n+\frac{4}{7}} + X_{217}\vartheta_{n+\frac{5}{7}} + X_{218}\vartheta_{n+\frac{6}{7}} + X_{219}\vartheta_{n+1}\right)\\
\rho''_{n-\frac{2}{7}} &= \rho''_n + h\left(X_{220}\vartheta_{n-\frac{1}{7}} + X_{221}\vartheta_{n-\frac{2}{7}} + X_{222}\vartheta_n + X_{223}\vartheta_{n+\frac{1}{7}} + X_{224}\vartheta_{n+\frac{2}{7}} + X_{225}\vartheta_{n+\frac{3}{7}} + X_{226}\vartheta_{n+\frac{4}{7}} + X_{227}\vartheta_{n+\frac{5}{7}} + X_{228}\vartheta_{n+\frac{6}{7}} + X_{229}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{1}{7}} &= \rho''_n + h\left(X_{230}\vartheta_{n-\frac{1}{7}} + X_{231}\vartheta_{n-\frac{2}{7}} + X_{232}\vartheta_n + X_{233}\vartheta_{n+\frac{1}{7}} + X_{234}\vartheta_{n+\frac{2}{7}} + X_{235}\vartheta_{n+\frac{3}{7}} + X_{236}\vartheta_{n+\frac{4}{7}} + X_{237}\vartheta_{n+\frac{5}{7}} + X_{238}\vartheta_{n+\frac{6}{7}} + X_{239}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{2}{7}} &= \rho''_n + h\left(X_{240}\vartheta_{n-\frac{1}{7}} + X_{241}\vartheta_{n-\frac{2}{7}} + X_{242}\vartheta_n + X_{243}\vartheta_{n+\frac{1}{7}} + X_{244}\vartheta_{n+\frac{2}{7}} + X_{245}\vartheta_{n+\frac{3}{7}} + X_{246}\vartheta_{n+\frac{4}{7}} + X_{247}\vartheta_{n+\frac{5}{7}} + X_{248}\vartheta_{n+\frac{6}{7}} + X_{249}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{3}{7}} &= \rho''_n + h\left(X_{250}\vartheta_{n-\frac{1}{7}} + X_{251}\vartheta_{n-\frac{2}{7}} + X_{252}\vartheta_n + X_{253}\vartheta_{n+\frac{1}{7}} + X_{254}\vartheta_{n+\frac{2}{7}} + X_{255}\vartheta_{n+\frac{3}{7}} + X_{256}\vartheta_{n+\frac{4}{7}} + X_{257}\vartheta_{n+\frac{5}{7}} + X_{258}\vartheta_{n+\frac{6}{7}} + X_{259}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{4}{7}} &= \rho''_n + h\left(X_{260}\vartheta_{n-\frac{1}{7}} + X_{261}\vartheta_{n-\frac{2}{7}} + X_{262}\vartheta_n + X_{263}\vartheta_{n+\frac{1}{7}} + X_{264}\vartheta_{n+\frac{2}{7}} + X_{265}\vartheta_{n+\frac{3}{7}} + X_{266}\vartheta_{n+\frac{4}{7}} + X_{267}\vartheta_{n+\frac{5}{7}} + X_{268}\vartheta_{n+\frac{6}{7}} + X_{269}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{5}{7}} &= \rho''_n + h\left(X_{270}\vartheta_{n-\frac{1}{7}} + X_{271}\vartheta_{n-\frac{2}{7}} + X_{272}\vartheta_n + X_{273}\vartheta_{n+\frac{1}{7}} + X_{274}\vartheta_{n+\frac{2}{7}} + X_{275}\vartheta_{n+\frac{3}{7}} + X_{276}\vartheta_{n+\frac{4}{7}} + X_{277}\vartheta_{n+\frac{5}{7}} + X_{278}\vartheta_{n+\frac{6}{7}} + X_{279}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{6}{7}} &= \rho''_n + h\left(X_{280}\vartheta_{n-\frac{1}{7}} + X_{281}\vartheta_{n-\frac{2}{7}} + X_{282}\vartheta_n + X_{283}\vartheta_{n+\frac{1}{7}} + X_{284}\vartheta_{n+\frac{2}{7}} + X_{285}\vartheta_{n+\frac{3}{7}} + X_{286}\vartheta_{n+\frac{4}{7}} + X_{287}\vartheta_{n+\frac{5}{7}} + X_{288}\vartheta_{n+\frac{6}{7}} + X_{289}\vartheta_{n+1}\right)\\
\rho''_{n+1} &= \rho''_n + h\left(X_{290}\vartheta_{n-\frac{1}{7}} + X_{291}\vartheta_{n-\frac{2}{7}} + X_{292}\vartheta_n + X_{293}\vartheta_{n+\frac{1}{7}} + X_{294}\vartheta_{n+\frac{2}{7}} + X_{295}\vartheta_{n+\frac{3}{7}} + X_{296}\vartheta_{n+\frac{4}{7}} + X_{297}\vartheta_{n+\frac{5}{7}} + X_{298}\vartheta_{n+\frac{6}{7}} + X_{299}\vartheta_{n+1}\right)
\end{aligned}\right\} \tag{11}$$

The unknown coefficient of $\Lambda$ in (9) can be seen in the appendix after simplifying $\Lambda_{\eta j} = \Psi^{-1}Z$.

Similarly, the unknown coefficients of the higher derivatives $X$in (10) and (11) are given in the appendix after simplification $X_{\eta j\sigma} = \Psi^{-1}E$.

## 3. Analysis of Basic Properties of New Numerical Method (NNM)

The basic properties of the NNM were analyzed, and the subsequent analysis led to the conclusion that these properties are order and error constant, consistency, zero-stability, convergent, and region of absolute stability.

### *3.1. Order and Error Constant*

Corollary 3.1 and Corollary 3.2 are used to obtain the order and error constant of NNM.

*Corollary 1 [19]*

The linear operator $L[\rho(\xi_n); h]$ associated with the local truncation error of the NNM defined in (9) to (11) is given as:

$C_{10}h^{10}\rho^{10}(\xi_n) + 0(h^{13}),\ C_{10}h^{10}\rho^{10}(\xi_n) + 0(h^{12}),\ C_{10}h^{10}\rho^{10}(\xi_n) + 0(h^{11}).$

*Proof*

Consider the linear difference operators associated with (9) to (11) are given by:

$$\left.\begin{aligned}
L[\rho(\xi_n);h] &= \rho_{n-\frac{1}{7}} - \rho_n + \frac{1}{7}h\rho'_n - \frac{\left(-\frac{1}{7}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{10}\vartheta_{n-\frac{1}{7}} + \Lambda_{11}\vartheta_{n-\frac{2}{7}} + \Lambda_{12}\vartheta_n + \Lambda_{13}\vartheta_{n+\frac{1}{7}} + \Lambda_{14}\vartheta_{n+\frac{2}{7}} + \Lambda_{15}\vartheta_{n+\frac{3}{7}} + \Lambda_{16}\vartheta_{n+\frac{4}{7}} + \Lambda_{17}\vartheta_{n+\frac{5}{7}} + \Lambda_{18}\vartheta_{n+\frac{6}{7}} + \Lambda_{19}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n-\frac{2}{7}} - \rho_n + \frac{2}{7}h\rho'_n - \frac{\left(-\frac{2}{7}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{20}\vartheta_{n-\frac{1}{7}} + \Lambda_{21}\vartheta_{n-\frac{2}{7}} + \Lambda_{22}\vartheta_n + \Lambda_{23}\vartheta_{n+\frac{1}{7}} + \Lambda_{24}\vartheta_{n+\frac{2}{7}} + \Lambda_{25}\vartheta_{n+\frac{3}{7}} + \Lambda_{26}\vartheta_{n+\frac{4}{7}} + \Lambda_{27}\vartheta_{n+\frac{5}{7}} + \Lambda_{28}\vartheta_{n+\frac{6}{7}} + \Lambda_{29}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+\frac{1}{7}} - \rho_n - \frac{1}{7}h\rho'_n - \frac{\left(\frac{1}{7}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{30}\vartheta_{n-\frac{1}{7}} + \Lambda_{31}\vartheta_{n-\frac{2}{7}} + \Lambda_{32}\vartheta_n + \Lambda_{33}\vartheta_{n+\frac{1}{7}} + \Lambda_{34}\vartheta_{n+\frac{2}{7}} + \Lambda_{35}\vartheta_{n+\frac{3}{7}} + \Lambda_{36}\vartheta_{n+\frac{4}{7}} + \Lambda_{37}\vartheta_{n+\frac{5}{7}} + \Lambda_{38}\vartheta_{n+\frac{6}{7}} + \Lambda_{39}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+\frac{2}{7}} - \rho_n - \frac{2}{7}h\rho'_n - \frac{\left(\frac{2}{7}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{40}\vartheta_{n-\frac{1}{7}} + \Lambda_{41}\vartheta_{n-\frac{2}{7}} + \Lambda_{42}\vartheta_n + \Lambda_{43}\vartheta_{n+\frac{1}{7}} + \Lambda_{44}\vartheta_{n+\frac{2}{7}} + \Lambda_{45}\vartheta_{n+\frac{3}{7}} + \Lambda_{46}\vartheta_{n+\frac{4}{7}} + \Lambda_{47}\vartheta_{n+\frac{5}{7}} + \Lambda_{48}\vartheta_{n+\frac{6}{7}} + \Lambda_{49}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+\frac{3}{7}} - \rho_n - \frac{3}{7}h\rho'_n - \frac{\left(\frac{3}{7}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{50}\vartheta_{n-\frac{1}{7}} + \Lambda_{51}\vartheta_{n-\frac{2}{7}} + \Lambda_{52}\vartheta_n + \Lambda_{53}\vartheta_{n+\frac{1}{7}} + \Lambda_{54}\vartheta_{n+\frac{2}{7}} + \Lambda_{55}\vartheta_{n+\frac{3}{7}} + \Lambda_{56}\vartheta_{n+\frac{4}{7}} + \Lambda_{57}\vartheta_{n+\frac{5}{7}} + \Lambda_{58}\vartheta_{n+\frac{6}{7}} + \Lambda_{59}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+\frac{4}{7}} - \rho_n - \frac{4}{7}h\rho'_n - \frac{\left(\frac{4}{7}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{60}\vartheta_{n-\frac{1}{7}} + \Lambda_{61}\vartheta_{n-\frac{2}{7}} + \Lambda_{62}\vartheta_n + \Lambda_{63}\vartheta_{n+\frac{1}{7}} + \Lambda_{64}\vartheta_{n+\frac{2}{7}} + \Lambda_{65}\vartheta_{n+\frac{3}{7}} + \Lambda_{66}\vartheta_{n+\frac{4}{7}} + \Lambda_{67}\vartheta_{n+\frac{5}{7}} + \Lambda_{68}\vartheta_{n+\frac{6}{7}} + \Lambda_{69}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+\frac{5}{7}} - \rho_n - \frac{5}{7}h\rho'_n - \frac{\left(\frac{5}{7}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{70}\vartheta_{n-\frac{1}{7}} + \Lambda_{71}\vartheta_{n-\frac{2}{7}} + \Lambda_{72}\vartheta_n + \Lambda_{73}\vartheta_{n+\frac{1}{7}} + \Lambda_{74}\vartheta_{n+\frac{2}{7}} + \Lambda_{75}\vartheta_{n+\frac{3}{7}} + \Lambda_{76}\vartheta_{n+\frac{4}{7}} + \Lambda_{77}\vartheta_{n+\frac{5}{7}} + \Lambda_{78}\vartheta_{n+\frac{6}{7}} + \Lambda_{79}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+\frac{6}{7}} - \rho_n - \frac{6}{7}h\rho'_n - \frac{\left(\frac{6}{7}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{80}\vartheta_{n-\frac{1}{7}} + \Lambda_{81}\vartheta_{n-\frac{2}{7}} + \Lambda_{82}\vartheta_n + \Lambda_{83}\vartheta_{n+\frac{1}{7}} + \Lambda_{84}\vartheta_{n+\frac{2}{7}} + \Lambda_{85}\vartheta_{n+\frac{3}{7}} + \Lambda_{86}\vartheta_{n+\frac{4}{7}} + \Lambda_{87}\vartheta_{n+\frac{5}{7}} + \Lambda_{88}\vartheta_{n+\frac{6}{7}} + \Lambda_{89}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+1} - \rho_n - h\rho'_n - \frac{(h)^2}{2!}\rho''_n - h^3\left(\Lambda_{90}\vartheta_{n-\frac{1}{7}} + \Lambda_{91}\vartheta_{n-\frac{2}{7}} + \Lambda_{92}\vartheta_n + \Lambda_{93}\vartheta_{n+\frac{1}{7}} + \Lambda_{94}\vartheta_{n+\frac{2}{7}} + \Lambda_{95}\vartheta_{n+\frac{3}{7}} + \Lambda_{96}\vartheta_{n+\frac{4}{7}} + \Lambda_{97}\vartheta_{n+\frac{5}{7}} + \Lambda_{98}\vartheta_{n+\frac{6}{7}} + \Lambda_{99}\vartheta_{n+1}\right)
\end{aligned}\right\}\tag{12}$$

$$\left.\begin{aligned}
L[\rho'(\xi_n;h] &= \rho'_{n-\frac{1}{7}} - \rho'_n + \frac{1}{7}h\rho''_n - h^2 f\left(\mathrm{X}_{110}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{111}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{112}\vartheta_n + \mathrm{X}_{113}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{114}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{115}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{116}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{117}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{118}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{119}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h] &= \rho'_{n-\frac{2}{7}} - \rho'_n + \frac{2}{7}h\rho''_n - h^2\left(\mathrm{X}_{120}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{121}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{122}\vartheta_n + \mathrm{X}_{123}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{124}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{125}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{126}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{127}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{128}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{129}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h] &= \rho'_{n+\frac{1}{7}} - \rho'_n - \frac{1}{7}h\rho''_n - h^2\left(\mathrm{X}_{130}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{131}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{132}\vartheta_n + \mathrm{X}_{133}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{134}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{135}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{136}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{137}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{138}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{139}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h] &= \rho'_{n+\frac{2}{7}} - \rho'_n - \frac{2}{7}h\rho''_n - h^2\left(\mathrm{X}_{140}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{141}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{142}\vartheta_n + \mathrm{X}_{143}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{144}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{145}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{146}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{147}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{148}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{149}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h] &= \rho'_{n+\frac{3}{7}} - \rho'_n - \frac{3}{7}h\rho''_n - h^2\left(\mathrm{X}_{150}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{151}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{152}\vartheta_n + \mathrm{X}_{153}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{154}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{155}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{156}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{157}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{158}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{159}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h] &= \rho'_{n+\frac{4}{7}} - \rho'_n - \frac{4}{7}h\rho''_n - h^2\left(\mathrm{X}_{160}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{161}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{162}\vartheta_n + \mathrm{X}_{163}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{164}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{165}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{166}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{167}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{168}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{169}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h] &= \rho'_{n+\frac{5}{7}} - \rho'_n - \frac{5}{7}h\rho''_n - h^2\left(\mathrm{X}_{170}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{171}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{172}\vartheta_n + \mathrm{X}_{173}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{174}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{175}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{176}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{177}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{178}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{179}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h] &= \rho'_{n+\frac{6}{7}} - \rho'_n - \frac{5}{7}h\rho''_n - h^2\left(\mathrm{X}_{180}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{181}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{182}\vartheta_n + \mathrm{X}_{183}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{184}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{185}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{186}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{187}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{188}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{189}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h] &= \rho'_{n+1} - \rho'_n - h\rho''_n - h^2\left(\mathrm{X}_{190}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{191}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{192}\vartheta_n + \mathrm{X}_{193}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{194}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{195}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{196}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{197}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{198}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{199}\vartheta_{n+1}\right)
\end{aligned}\right\}\tag{13}$$

$$
\left.\begin{aligned}
L[\rho''(\xi_n);h] &= \rho''_{n-\frac{1}{7}} - \rho''_n - h\left(\mathrm{X}_{210}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{211}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{212}\vartheta_n + \mathrm{X}_{213}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{214}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{215}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{216}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{217}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{218}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{219}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h] &= \rho''_{n-\frac{2}{7}} - \rho''_n - h\left(\mathrm{X}_{220}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{221}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{222}\vartheta_n + \mathrm{X}_{223}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{224}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{225}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{226}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{227}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{228}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{229}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h] &= \rho''_{n+\frac{1}{7}} - \rho''_n - h\left(\mathrm{X}_{230}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{231}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{232}\vartheta_n + \mathrm{X}_{233}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{234}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{235}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{236}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{237}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{238}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{239}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h] &= \rho''_{n+\frac{2}{7}} - \rho''_n - h\left(\mathrm{X}_{240}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{241}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{242}\vartheta_n + \mathrm{X}_{243}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{244}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{245}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{246}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{247}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{248}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{249}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h] &= \rho''_{n+\frac{3}{7}} - \rho''_n - h\left(\mathrm{X}_{250}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{251}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{252}\vartheta_n + \mathrm{X}_{253}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{254}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{255}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{256}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{257}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{258}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{259}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h] &= \rho''_{n+\frac{4}{7}} - \rho''_n - h\left(\mathrm{X}_{260}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{261}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{262}\vartheta_n + \mathrm{X}_{263}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{264}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{265}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{266}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{267}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{268}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{269}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h] &= \rho''_{n+\frac{5}{7}} - \rho''_n - h\left(\mathrm{X}_{270}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{271}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{272}\vartheta_n + \mathrm{X}_{273}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{274}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{275}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{276}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{277}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{278}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{279}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h] &= \rho''_{n+\frac{6}{7}} - \rho''_n - h\left(\mathrm{X}_{280}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{281}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{282}\vartheta_n + \mathrm{X}_{283}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{284}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{285}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{286}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{287}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{288}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{289}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h] &= \rho''_{n+1} - \rho''_n - h\left(\mathrm{X}_{290}\vartheta_{n-\frac{1}{7}} + \mathrm{X}_{291}\vartheta_{n-\frac{2}{7}} + \mathrm{X}_{292}\vartheta_n + \mathrm{X}_{293}\vartheta_{n+\frac{1}{7}} + \mathrm{X}_{294}\vartheta_{n+\frac{2}{7}} + \mathrm{X}_{295}\vartheta_{n+\frac{3}{7}} + \mathrm{X}_{296}\vartheta_{n+\frac{4}{7}} + \mathrm{X}_{297}\vartheta_{n+\frac{5}{7}} + \mathrm{X}_{298}\vartheta_{n+\frac{6}{7}} + \mathrm{X}_{299}\vartheta_{n+1}\right)
\end{aligned}\right\} \tag{14}
$$

*Corollary 2 [19]*

The local truncation error of (9) to (11) is assumed $\rho(\xi)$ to be sufficiently differentiable, and expanding equations (12) to (14) $\xi_n$ using a Taylor series to obtain

$$
\begin{aligned}
&L_{-\frac{1}{7}}[\rho(\xi_n);h] = (-1.3423\times10^{-15}),\ L_{-\frac{2}{7}}[\rho(\xi_n);h] = (-4.9363\times10^{-15}),\ L_{\frac{1}{7}}[\rho(\xi_n);h] = (-6.2566\times10^{-16}),\\
&L_{\frac{2}{7}}[\rho(\xi_n);h] = (-4.1514\times10^{-15}),\ L_{\frac{3}{7}}[\rho(\xi_n);h] = (-9.9180\times10^{-15}),\ L_{\frac{4}{7}}[\rho(\xi_n);h] = (-1.8499\times10^{-14}),\\
&L_{\frac{5}{7}}[\rho(\xi_n);h] = (-2.9236\times10^{-14}),\ L_{\frac{6}{7}}[\rho(\xi_n);h] = (-4.3060\times10^{-14}),\ L_{1}[\rho(\xi_n);h] = (-5.9688\times10^{-14}),
\end{aligned}
$$

$$
\begin{aligned}
&L_{-\frac{1}{7}}[\rho'(\xi_n);h] = (3.4723\times10^{-14}),\ L_{-\frac{2}{7}}[\rho'(\xi_n);h] = (-7.9947\times10^{-14}),\ L_{\frac{1}{7}}[\rho'(\xi_n);h] = (-1.3948\times10^{-09}),\\
&L_{\frac{2}{7}}[\rho'(\xi_n);h] = (-3.2647\times10^{-14}),\ L_{\frac{3}{7}}[\rho'(\xi_n);h] = (-5.0341\times10^{-14}),\ L_{\frac{4}{7}}[\rho'(\xi_n);h] = (-6.7031\times10^{-14}),\\
&L_{\frac{5}{7}}[\rho'(\xi_n);h] = (-8.8471\times10^{-14}),\ L_{\frac{6}{7}}[\rho'(\xi_n);h] = (-8.9137\times10^{-14}),\ L_{1}[\rho'(\xi_n);h] = (-2.3919\times10^{-13}),
\end{aligned}
$$

$$
\begin{aligned}
&L_{-\frac{1}{7}}[\rho''(\xi_n);h] = (-5.5970\times10^{-13}),\ L_{-\frac{2}{7}}[\rho''(\xi_n);h] = (2.8721\times10^{-12}),\ L_{\frac{1}{7}}[\rho''(\xi_n);h] = (-1.8069\times10^{-13}),\\
&L_{\frac{2}{7}}[\rho''(\xi_n);h] = (-8.4802\times10^{-14}),\ L_{\frac{3}{7}}[\rho''(\xi_n);h] = (-1.6292\times10^{-13}),\ L_{\frac{4}{7}}[\rho''(\xi_n);h] = (-6.7031\times10^{-14}),\\
&L_{\frac{5}{7}}[\rho''(\xi_n);h] = (-2.4772\times10^{-13}),\ L_{\frac{6}{7}}[\rho''(\xi_n);h] = (3.1198\times10^{-13}),\ L_{1}[\rho''(\xi_n);h] = (-3.1199\times10^{-12})
\end{aligned}
$$

*Proof*

Simplify equations (9) to (11) with Corollary 2 and collect the like terms to obtain:

$$
\begin{aligned}
&L_{-\frac{1}{7}}[\rho(\xi_n);h] = (-1.3423\times10^{-15})C_{10}h^{10}\rho^{10}(\xi_n) + 0(h^{13}),\ L_{-\frac{2}{7}}[\rho(\xi_n);h] = (-4.9363\times10^{-15})C_{10}h^{10}\rho^{10}(\xi_n) + 0(h^{13}),\\
&L_{\frac{1}{7}}[\rho(\xi_n);h] = (-6.2566\times10^{-16})C_{10}h^{10}\rho^{10}(\xi_n) + 0(h^{13}),\ L_{\frac{2}{7}}[\rho(\xi_n);h] = (-4.1514\times10^{-15})C_{10}h^{10}\rho^{10}(\xi_n) + 0(h^{13}),\\
&L_{\frac{3}{7}}[\rho(\xi_n);h] = (-9.9180\times10^{-15})C_{10}h^{10}\rho^{10}(\xi_n) + 0(h^{13}),\ L_{\frac{4}{7}}[\rho(\xi_n);h] = (-1.8499\times10^{-14})C_{10}h^{10}\rho^{10}(\xi_n) + 0(h^{13}),\\
&L_{\frac{5}{7}}[\rho(\xi_n);h] = (-2.9236\times10^{-14})C_{10}h^{10}\rho^{10}(\xi_n) + 0(h^{13}),\ L_{\frac{6}{7}}[\rho(\xi_n);h] = (-4.3060\times10^{-14})C_{10}h^{10}\rho^{10}(\xi_n) + 0(h^{13}),\\
&L_{1}[\rho(\xi_n);h] = (-5.9688\times10^{-14})C_{10}h^{10}\rho^{10}(\xi_n) + 0(h^{13}),
\end{aligned}
$$

$$L_{-\frac{1}{7}}[\rho'(\xi_n);h]=(3.4723\times10^{-14})C_{10}h^{10}\rho^{10}(\xi_n)+0(h^{12}),\ L_{-\frac{2}{7}}[\rho'(\xi_n);h]=(-7.9947\times10^{-14})C_{10}h^{10}\rho^{10}(\xi_n)+0(h^{12}),$$
$$L_{\frac{1}{7}}[\rho'(\xi_n);h]=(-1.3948\times10^{-09})C_{10}h^{10}\rho^{10}(\xi_n)+0(h^{12}),\ L_{\frac{2}{7}}[\rho'(\xi_n);h]=(-3.2647\times10^{-14})C_{10}h^{10}\rho^{10}(\xi_n)+0(h^{12}),$$
$$L_{\frac{3}{7}}[\rho'(\xi_n);h]=(-5.0341\times10^{-14})C_{10}h^{10}\rho^{10}(\xi_n)+0(h^{12}),\ L_{\frac{4}{7}}[\rho'(\xi_n);h]=(-6.7031\times10^{-14})C_{10}h^{10}\rho^{10}(\xi_n)+0(h^{12}),$$
$$L_{\frac{5}{7}}[\rho'(\xi_n);h]=(-8.8471\times10^{-14})C_{10}h^{10}\rho^{10}(\xi_n)+0(h^{12}),\ L_{\frac{6}{7}}[\rho'(\xi_n);h]=(-8.9137\times10^{-14})C_{10}h^{10}\rho^{10}(\xi_n)+0(h^{12}),$$
$$L_{1}[\rho'(\xi_n);h]=(-2.3919\times10^{-13})C_{10}h^{10}\rho^{10}(\xi_n)+0(h^{12}),$$

$$L_{-\frac{1}{7}}[\rho''(\xi_n);h]=(-5.5970\times10^{-13})_{10}h^{10}\rho^{10}(\xi_n)+0(h^{11}),\ L_{-\frac{2}{7}}[\rho''(\xi_n);h]=(2.8721\times10^{-12})_{10}h^{10}\rho^{10}(\xi_n)+0(h^{11}),$$
$$L_{\frac{1}{7}}[\rho''(\xi_n);h]=(-1.8069\times10^{-13})_{10}h^{10}\rho^{10}(\xi_n)+0(h^{11}),\ L_{\frac{2}{7}}[\rho''(\xi_n);h]=(-8.4802\times10^{-14})_{10}h^{10}\rho^{10}(\xi_n)+0(h^{11}),$$
$$L_{\frac{3}{7}}[\rho''(\xi_n);h]=(-1.6292\times10^{-13})_{10}h^{10}\rho^{10}(\xi_n)+0(h^{11}),\ L_{\frac{4}{7}}[\rho''(\xi_n);h]=(-6.7031\times10^{-14})_{10}h^{10}\rho^{10}(\xi_n)+0(h^{11}),$$
$$L_{\frac{5}{7}}[\rho''(\xi_n);h]=(-2.4772\times10^{-13})_{10}h^{10}\rho^{10}(\xi_n)+0(h^{11}),\ L_{\frac{6}{7}}[\rho''(\xi_n);h]=(3.1198\times10^{-13})_{10}h^{10}\rho^{10}(\xi_n)+0(h^{11}),$$
$$L_{1}[\rho''(\xi_n);h]=(-3.1199\times10^{-12})_{10}h^{10}\rho^{10}(\xi_n)+0(h^{11})$$

### 3.2. *Consistency*

Definition 2: First and Second Characteristic Polynomials

Given the NNM, the first and second characteristic polynomials are defined as,

$$\rho(z)=\sum_{j=0}^{1}\alpha_j z^j \tag{15}$$

and

$$\sigma(z)=\sum_{j=0}^{1}\beta_j z^j \tag{16}$$

where $z$ is the principal root, $\alpha_1\neq0$ and $\alpha_0^2+\beta_0^2\neq0$ [19].

The new numerical scheme is said to be consistent if it satisfies the following conditions;

i. the order $p\geq1$,

ii. $\sum_{j=0}^{1}\alpha_j=0$ and

iii. $p'(1)=\sigma(1)$

The NNM is consistent since it is of uniform order ten (According to definition 2).

### 3.3. *Zero Stability*

Definition 3: A new NNM is said to be zero-stable if the roots $z_s,\ s=1,2,\dots,n$ of the first characteristic polynomial $\overline{p}(z)$, defined by

$$\overline{p}(z)=det[zA^{(0)}-E] \tag{17}$$

satisfies $|z_s| \leq 1$ and every root $|z_s| = 1$ has multiplicity not exceeding the order of the differential equation as $h \to 0$. Moreover, as $h \to 0$, $p(z) = z^{r-\mu}(z-1)^{\mu}$, where $\mu$ is the order of the differential equation, $r$ is the order of the matrices $A^{(0)}$ *and* $E$. The main consequence of zero-stability is to control the propagation of the error as the integration proceeds [19].

By definition 3, a NNM is said to be Zero-stable for any well-behaved problem, that is

$$\rho(u) = \frac{32091-1393u+18230u^2+51194u^3\text{-}21346u^4+87627u^5-89621u^6-7112u^7+1067u^8}{21109-34812u+98217u^2-91276u^3\text{-}124576u^4+9021u^5+1061u^6-990u^7+701u^8+89u^9} \tag{18}$$

Thus, solving for $z$ in

$$\rho(z) = \left| z\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix} - \begin{bmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1 \end{bmatrix} \right| = \begin{bmatrix} z&0&0&0&0&0&0&-1\\ 0&z&0&0&0&0&0&-1\\ 0&0&z&0&0&0&0&-1\\ 0&0&0&z&0&0&0&-1\\ 0&0&0&0&z&0&0&-1\\ 0&0&0&0&0&z&0&-1\\ 0&0&0&0&0&0&z&-1\\ 0&0&0&0&0&0&0&z-1 \end{bmatrix} = z^7(z-1)$$

$$z^7(z-1) = 0 \tag{19}$$

Solving for (19) gives $z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = z_7 = 0$ and $z_8 = 1$. Hence, the NNM is zero-stable.

### *3.4. Convergence*

The NNM is convergent, since it is consistent and zero-stable by a theorem that states "the necessary and sufficient conditions for NNM to be convergent are that it must be consistent and zero-stable [19].

### *3.5. Region of Absolute Stability (RAS)*

In the quest for the new LBA's absolute stability regions, a technique was utilized that neither necessitated the finding of polynomial roots nor the solving of multiple inequalities [19]. This process is referred to as the Boundary Locus Method (BLM). The BLM was implemented on NNM for the purpose of generating the stability polynomial of the kind.

$$\left.\begin{aligned} &\bar{h}(\pi) = \left(1.28681378\ 7395547140\ 30\times10^{-10}\pi^8 + 1.64274957\ 1204878248\ 10\times10^{-11}w^9\right)h^9 \\ &+\left(3.12484218\ 4569379614\ 70\times10^{-09}\pi^8 + 1.09584307\ 6645631702\ 50\times10^{-09}w^9\right)h^8 \\ &\left(-2.93861891\ 7813720837\ 10\times10^{08}\pi^9 - 6.12109442\ 6077876277\ 50\times10^{-08}\pi^8\right)h^7 \\ &+\left(-3.57085354\ 5916647297\ 30\times10^{-07}\pi^9 - 0.00000223\ 6010425800\ 23\pi^8\right)h^6 \\ &+\left(0.00001615\ 6837402580\ 06\pi^9 + 0.00000343\ 1660270194\ 01\pi^8\right)h^5 + \\ &\left(0.00067180\ 0491387127\ 78\pi^8 - 0.00006213\ 3276297174\ 21\pi^9\right)h^4 \\ &+\left(-0.00330853\ 2236692052\ 91\pi^9 + 0.00568518\ 0633665233\ 18\pi^8\right)h^3 + \\ &\left(0.05497730\ 7491638000\ 63\pi^9 - 0.01918674\ 6897582138\ 49\pi^8\right)h^2 \\ &+\left(-0.35714285\ 7142857142\ 86\pi^9 - 0.35714285\ 7142857142\ 86\pi^8\right)h - \pi^8 + \pi^9 \end{aligned}\right\} \tag{20}$$

The RAS of NNM was obtained by using the stability polynomial (20) on MATLAB R2024a as:

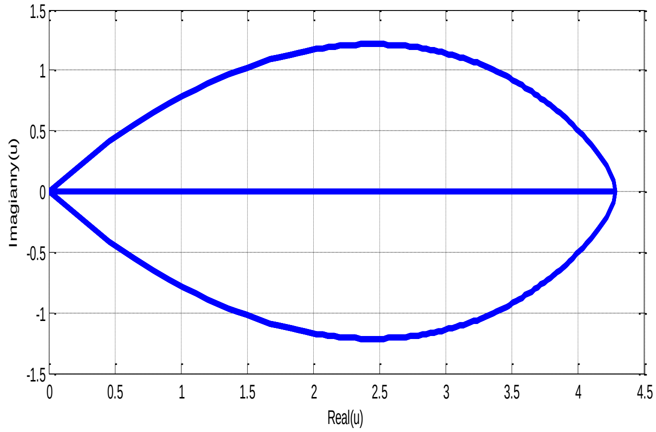


**Fig. 1 Regions of absolute stability**

## 4. Numerical Application of New Numerical Schemes (NNM)

The NNM was applied to (VIDEs) of the second kind, where the results were numerically tabulated and textually shown for comparison with existing methods in terms of error. The numerical application of the NNM in this case was done by solving new numerical schemes for Volterra Integro-Differential Equations (VIDEs) of the second kind. The following acronyms are used throughout the tables, figures, discussions of results, and other sections of this work and were specifically used in the tables and figures for comparisons. PE denotes Points of Evaluation, AS represents the Analytic Solution, CNR refers to the Computed Numerical Results obtained using the new LBA, and ENM indicates the Error in the NNM. TM stands for the Trapezoidal Method as presented in [17], while ETM denotes the Extended Trapezoidal Method implemented in [17], and NETM refers to the Numerical Extended Trapezoidal Method of [22]. EABM5 represents the error in the Fifth-Order Adams–Bashforth–Moulton predictor–corrector method developed by [20], and E2P3B denotes the error in the Two-Point Three-Step Block Method developed by [21]. Furthermore, CMM refers to the Continuous Multistep Method of [23], 4SBM represents the Four-Step Block Method developed by [23], and 5SBM denotes the Five-Step Block Method developed by [23].

Example 1: Give it a thought regarding the Volterra integro-differential second kind equation, which reads:

$$\rho'(\xi) = 1 - \int_0^{\xi} \rho(\tau)d\tau,\ \rho(0) = 0,\ 0 \le \xi \le 1 \tag{21}$$

with an analytic solution of the form

$$\rho(\xi) = sin(\xi) \tag{22}$$

Source: [15, 20-22].

Example 2: Give it a thought regarding the Volterra integro-differential second kind equation, which reads:

$$\rho''(\xi) + \int_0^{\xi} \left(\rho(\tau)\right)^2 d\tau + \left(\frac{\xi}{2} - sinh(\xi) - \frac{1}{4} sinh(2\xi)\right) = 0,\ \rho(0) = 0,\ \rho'(0) = 1 \tag{23}$$

with an analytic solution of the form

$$\rho(\xi) = sinh(\xi) \tag{24}$$

Source: [23].

**Table 1. Comparison of numerical results for example 1 when $h = 0.01$**

| $\xi$ | Approximate Solution | Numerical Solution | ENM | ETM | EETM | ENETM |
|---|---|---|---|---|---|---|
| 0.01 | 0.00999983333416666468 | 0.00999983333416666468 | 0.0000e00 | 8.2978e-07 | 8.2916e-07 | 1.2051e-09 |
| 0.02 | 0.01999866669333307937 | 0.01999866669333307937 | 0.0000e00 | 1.6359e-06 | 1.6334e-06 | 1.0290e-08 |
| 0.03 | 0.02999550020249566077 | 0.02999550020249566077 | 0.0000e00 | 2.3938e-06 | 2.3883e-06 | 3.5392e-08 |
| 0.04 | 0.03998933418663415945 | 0.03998933418663415945 | 0.0000e00 | 3.0799e-06 | 3.0702e-06 | 8.4328e-08 |
| 0.05 | 0.04997916927067832880 | 0.04997916927067832880 | 0.0000e00 | 3.6715e-06 | 3.6565e-06 | 1.6443e-07 |
| 0.06 | 0.05996400647944459920 | 0.05996400647944459920 | 0.0000e00 | 4.1478e-06 | 4.1266e-06 | 2.8242e-07 |
| 0.07 | 0.06994284733753276398 | 0.06994284733753276398 | 0.0000e00 | 4.4897e-06 | 4.4615e-06 | 4.4421e-07 |
| 0.08 | 0.07991469396917268731 | 0.07991469396917268731 | 0.0000e00 | 4.6805e-06 | 4.6447e-06 | 6.5484e-07 |
| 0.09 | 0.08987854919801104969 | 0.08987854919801104969 | 0.0000e00 | 4.7061e-06 | 4.6620e-06 | 9.1831e-07 |
| 0.10 | 0.09983341664682815231 | 0.09983341664682815231 | 0.0000e00 | 4.5551e-06 | 4.5025e-06 | 1.2375e-06 |

**Table 2. Comparison of numerical results for example 1 when $h = 0.1$**

| $\xi$ | Approximate Solution | Numerical Solution | ENM | ETM | EETM | ENETM |
|---|---|---|---|---|---|---|
| 0.100 | 0.09983341664682815231 | 0.09983341664682815231 | 0.0000e00 | 8.4317e-05 | 8.2792e-05 | 1.2502e-07 |
| 0.200 | 0.19866933079506121546 | 0.19866933079506121546 | 0.0000e00 | 1.6558e-04 | 1.6310e-04 | 7.301e5-07 |
| 0.300 | 0.29552020666133957511 | 0.29552020666133957512 | 1.0000e-20 | 2.4402e-04 | 2.3848e-04 | 2.5669e-06 |
| 0.400 | 0.38941834230865049167 | 0.38941834230865049225 | 5.8000e-19 | 3.1661e-04 | 3.0657e-04 | 6.4301e-06 |
| 0.500 | 0.47942553860420300027 | 0.47942553860420301081 | 1.0540e-17 | 3.8012e-04 | 3.6514e-04 | 1.3074e-05 |
| 0.600 | 0.56464247339503535720 | 0.56464247339503546869 | 1.1149e-16 | 4.3328e-04 | 4.1210e-04 | 2.3195e-05 |
| 0.700 | 0.64421768723769105367 | 0.64421768723769186988 | 8.1621e-16 | 4.7376e-04 | 4.4558e-04 | 3.7422e-05 |
| 0.800 | 0.71735609089952276163 | 0.71735609089952732283 | 4.5612e-15 | 4.9978e-04 | 4.6392e-04 | 5.6296e-05 |
| 0.900 | 0.78332690962748338846 | 0.78332690962750411715 | 2.0729e-14 | 5.0976e-04 | 4.6571e-04 | 8.0264e-05 |
| 1.000 | 0.84147098480789650665 | 0.84147098480797650132 | 7.9996e-14 | 5.0243e-04 | 4.4985e-04 | 1.0966e-04 |

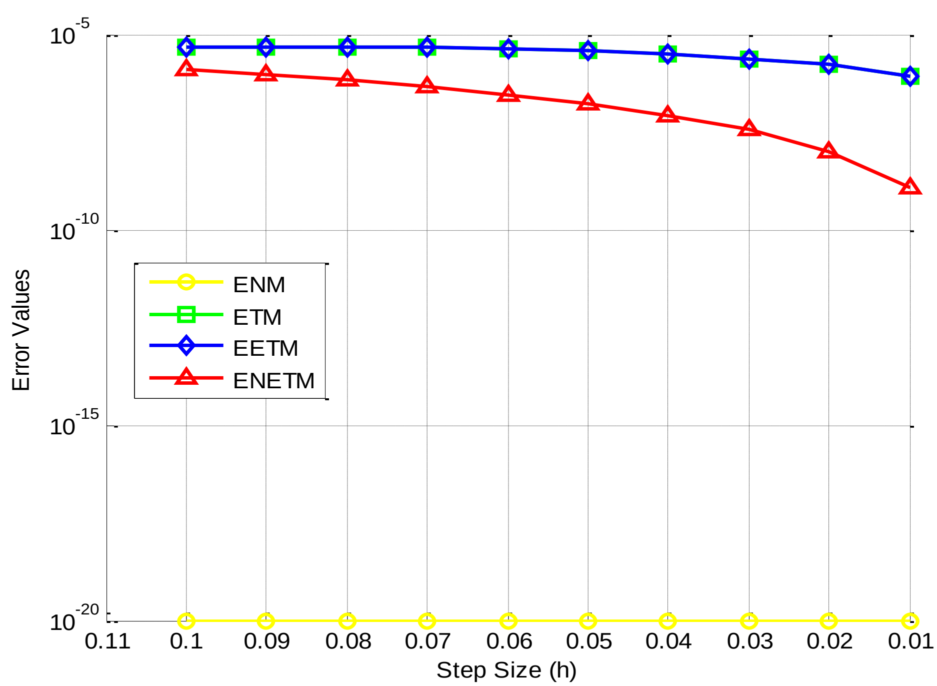


**Fig. 2 Comparison of numerical results for example 1 when $h = 0.01$**

**Table 3. Comparison of numerical results for example 2**

| $\xi$ | Approximate Solution | Numerical Solution | ENM | ECMM | E4SBM | E5SBM |
|---|---|---|---|---|---|---|
| 0.16 | 0.16068354101279944828 | 0.16068354101279944820 | 8.0000e-20 | 4.5700e-08 | 4.5800e-08 | 4.5700e-08 |
| 0.32 | 0.32548936363113307984 | 0.32548936363113307989 | 5.0000e-20 | 3.7140e-07 | 3.7160e-07 | 3.7150e-07 |
| 0.48 | 0.49864550519337626463 | 0.49864550519337626442 | 2.1000e-19 | 1.2858e-06 | 1.2861e-06 | 1.2859e-06 |
| 0.64 | 0.68459422763095139805 | 0.88810598218762308193 | 1.1600e-18 | 3.1554e-06 | 3.1559e-06 | 3.1555e-06 |
| 0.80 | 0.88810598218762300658 | 0.88810598218762308193 | 7.5350e-17 | 6.4379e-06 | 6.4386e-06 | 6.4381e-06 |
| 0.96 | 1.11440179372400284780 | 1.11440179372400229834 | 5.4950e-16 | 1.1717e-05 | 1.1718e-05 | 1.1718e-05 |

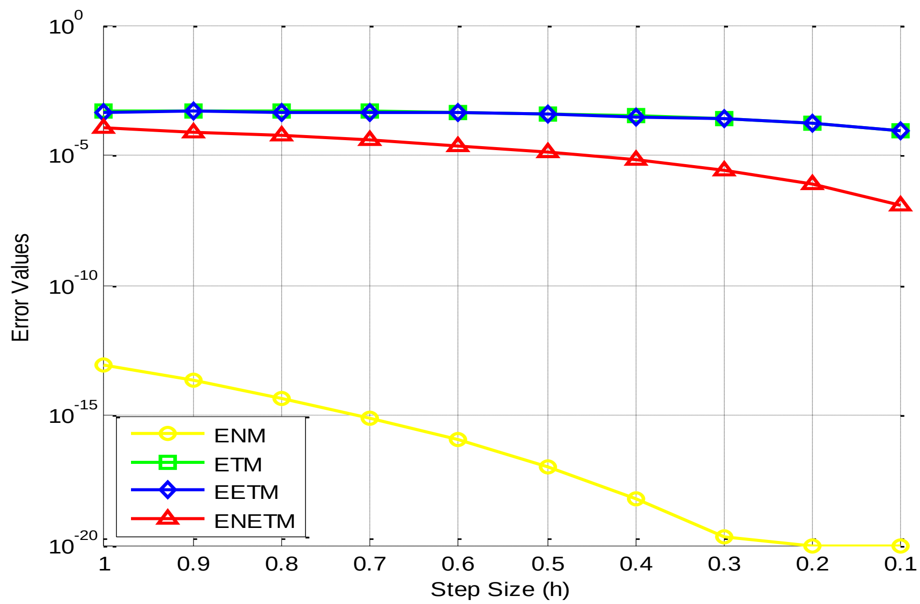


**Fig. 3 Comparison of numerical results for example 1 when $h = 0.1$**

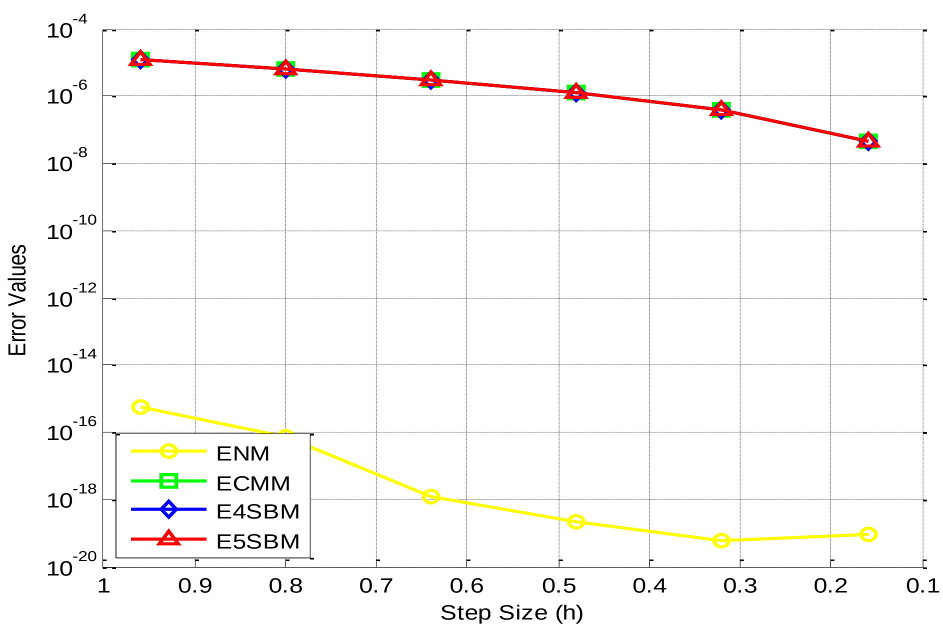


**Fig. 4 Comparison of numerical results for example 2**

## 5. Discussion of Results

The New Numerical Method (NNM) originated through the combination of a linear block algorithm that solved the third derivative together with the numerical methods used for second-kind Volterra integro-differential equations described in Proposition 1 through its transformation into a Linear Block Algorithm, which followed the techniques existing in [18]. A hybrid block framework consisting of higher derivatives was created from a standard linear multistep method, leading to continuous polynomial schemes and their discrete expansions, which produced the final NNM with unknown coefficients that the appendix presents in analytical form. The core properties of the method show a complete examination, which demonstrates that NNM delivers excellent numerical performance because it provides an order definition together with an error constant and maintains zero-stability while achieving convergence and establishing an absolute stability region. The order and local truncation error were determined through the use of the related linear difference operators and Taylor series expansions, verifying that the method achieves a uniform order of ten. The characteristic polynomial conditions require fulfillment to establish consistency, while zero-stability was checked with the roots of the first characteristic polynomial, thereby making sure that error propagation was under control. The method achievements combine with zero-stability to establish convergence because the Boundary Locus Method successfully determined the absolute stability region, and the graphical representation proved that the method works for both stiff and well-behaved problems.

The data outcomes indicated in Tables 1 and 2 demonstrate the NNM solution for the second kind Volterra integro-differential equations through its application to Example 1. The new scheme produced numerical results that matched the analytical solution at every evaluation point, resulting in NNM errors that were either nonexistent or extremely small. The Trapezoidal, Extended Trapezoidal, and Numerical Extended Trapezoidal methods generate traditional results that show more errors that increase when users select larger step sizes or additional evaluation points. The steady reduction of errors demonstrates that NNM provides better accuracy and reliability than all other methods tested in this research.

The NNM proves its stable performance through both its small and large step size testing, which shows better results than previous methods. The existing methods suffer a considerable increase in errors as the interval gets larger, while the NNM maintains errors that are either completely zero or extremely small, even at the highest points of evaluation. The method exhibits this property because it demonstrates strong convergence capabilities and maintains numerical stability throughout its execution. The results show that NNM provides effective error control, which makes it suitable for solving second-kind VIDEs across extended computational periods.

The findings of Example 2, which Table 3 presents, establish stronger proof for the earlier findings. The NNM generates numerical solutions that show high accuracy with the analytic solution, while its associated errors remain lower than the errors produced by the Continuous Multistep Method, Four-Step Block Method, and Five-Step Block Method. The results are also depicted in Figures 1–4, which confirm very clearly that NNM solutions are practically indistinguishable from analytic solutions, whereas the other methods can only show very slight differences. The results presented in tables and graphs evaluated NNM as an accurate and consistent method that works efficiently to solve second-kind Volterra integro-differential equations, making it a competitive solution against its rivals.

## 6. Summary and Conclusion

The study created a new Equally Spaced Numerical Method (NNM), which solves Volterra integro-differential equations of the second kind through numerical methods. The hybrid block scheme was developed through the combination of a third-derivative linear block algorithm and a linear multistep method, which operated on equally spaced points. The derivation produced continuous polynomial approximations and their discrete counterparts, which contained analytically determined coefficients. The NNM was thoroughly examined through theoretical studies, which included tests for order and error constant, consistency, zero-stability, convergence, and the region of absolute stability. The analysis confirmed that the method maintains a continuous order of ten while meeting all fundamental criteria required for effective numerical methods. The researchers assessed NNM performance through Volterra integro-differential equations of the second kind, which they solved using multiple numerical examples. The results presented in tables and graphs demonstrated that the NNM produces numerical solutions that match analytic solutions with high accuracy because their evaluation points show either minimal or nonexistent errors. The NNM showed superior accuracy and error management while maintaining numerical stability throughout both short and extended computational periods when compared to traditional trapezoidal methods, block methods, and other techniques.

The researchers developed a new Equally Spaced Numerical Method and conducted testing to solve Volterra integro-differential equations of the second kind. The theoretical analysis proved the method to be consistent, zero-stable, and convergent with a broad region of absolute stability, which is typical of methods that can handle both regular and stiff mathematical problems. The numerical tests verified the claim that the new method is more accurate and efficient than several existing methods, with significantly lower error values and almost absolute agreement with analytic solutions. The NNM functions as a fast and accurate numerical solution method for second-kind Volterra integro-differential equations, which also enables researchers to develop more advanced methods for various integro-differential and functional equations.

$$\begin{pmatrix} X_{210} \\ X_{211} \\ X_{212} \\ X_{213} \\ X_{214} \\ X_{215} \\ X_{216} \\ X_{217} \\ X_{218} \\ X_{219} \end{pmatrix} = \begin{pmatrix} -\frac{2655563}{50803200} \\ \frac{1169}{1036800} \\ -\frac{859009}{6350400} \\ \frac{274849}{3175200} \\ -\frac{397331}{5080320} \\ -\frac{1424417}{25401600} \\ \frac{92567}{3175200} \\ \frac{65039.}{6350400} \\ \frac{110219}{50803200} \\ \frac{425}{2032128} \end{pmatrix}, \begin{pmatrix} X_{220} \\ X_{221} \\ X_{222} \\ X_{223} \\ X_{224} \\ X_{225} \\ X_{226} \\ X_{227} \\ X_{228} \\ X_{229} \end{pmatrix} = \begin{pmatrix} -\frac{37829}{158760} \\ -\frac{3956}{99225} \\ \frac{34369}{396900} \\ -\frac{45331}{198450} \\ \frac{103987}{396900} \\ -\frac{83291}{396900} \\ \frac{9239}{79380} \\ -\frac{8467}{198450} \\ \frac{3697}{396900} \\ -\frac{9}{9800} \end{pmatrix}, \begin{pmatrix} X_{230} \\ X_{231} \\ X_{232} \\ X_{233} \\ X_{234} \\ X_{235} \\ X_{236} \\ X_{237} \\ X_{238} \\ X_{229} \end{pmatrix} = \begin{pmatrix} \frac{163531}{50803200} \\ \frac{425}{2032128} \\ \frac{391711}{6350400} \\ \frac{349817}{3175200} \\ \frac{1083167}{25401600} \\ \frac{129581}{5080320} \\ -\frac{38599}{3175200} \\ \frac{25759}{6350400} \\ \frac{42187}{50803200} \\ \frac{1}{12800} \end{pmatrix},$$

$$\begin{pmatrix} X_{240} \\ X_{241} \\ X_{242} \\ X_{243} \\ X_{244} \\ X_{245} \\ X_{246} \\ X_{247} \\ X_{248} \end{pmatrix} = \begin{pmatrix} -\frac{1769}{793800} \\ \frac{13}{99225} \\ \frac{21809}{396900} \\ \frac{7193}{39690} \\ \frac{20291}{396900} \\ \frac{1013}{396900} \\ \frac{1213}{396900} \\ \frac{253}{198450} \\ \frac{23}{79380} \\ \frac{23}{793800} \end{pmatrix}, \begin{pmatrix} X_{250} \\ X_{251} \\ X_{252} \\ X_{253} \\ X_{254} \\ X_{255} \\ X_{256} \\ X_{256} \\ X_{258} \\ X_{259} \end{pmatrix} = \begin{pmatrix} \frac{351}{125440} \\ \frac{113}{627200} \\ \frac{4559}{78400} \\ \frac{6609}{39200} \\ -\frac{41553}{313600} \\ \frac{26321}{313600} \\ -\frac{123}{7840} \\ \frac{351}{78400} \\ \frac{11}{12800} \\ \frac{1}{12800} \end{pmatrix}, \begin{pmatrix} X_{260} \\ X_{261} \\ X_{262} \\ X_{263} \\ X_{264} \\ X_{265} \\ X_{266} \\ X_{267} \\ X_{268} \\ X_{269} \end{pmatrix} = \begin{pmatrix} \frac{23}{14175} \\ \frac{13}{99225} \\ \frac{5494}{99225} \\ \frac{17632}{99225} \\ \frac{2174}{19845} \\ \frac{17632}{99225} \\ -\frac{5494}{99225} \\ \frac{32}{14175} \\ \frac{13}{99225} \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} X_{270} \\ X_{271} \\ X_{272} \\ X_{273} \\ X_{274} \\ X_{275} \\ X_{276} \\ X_{277} \\ X_{278} \\ X_{279} \end{pmatrix} = \begin{pmatrix} -\frac{6275}{2032128} \\ \frac{425}{2032128} \\ \frac{15095}{254016} \\ \frac{21025}{127008} \\ \frac{137225}{1016064} \\ \frac{137225}{1016064} \\ \frac{21025}{127008} \\ \frac{15095}{254016} \\ -\frac{6275}{2032128} \\ \frac{425}{2032128} \end{pmatrix}, \begin{pmatrix} X_{280} \\ X_{281} \\ X_{282} \\ X_{283} \\ X_{284} \\ X_{285} \\ X_{286} \\ X_{287} \\ X_{288} \\ X_{289} \end{pmatrix} = \begin{pmatrix} -\frac{9}{9800} \\ 0 \\ \frac{241}{4900} \\ \frac{477}{2450} \\ \frac{387}{4900} \\ \frac{209}{980} \\ \frac{387}{4900} \\ \frac{477}{2450} \\ \frac{241}{4900} \\ -\frac{9}{9800} \end{pmatrix}, \begin{pmatrix} X_{290} \\ X_{291} \\ X_{292} \\ X_{293} \\ X_{294} \\ X_{285} \\ X_{296} \\ X_{297} \\ X_{298} \\ X_{299} \end{pmatrix} = \begin{pmatrix} -\frac{12859}{1036800} \\ \frac{1169}{1036800} \\ \frac{13231}{129600} \\ \frac{637}{12960} \\ \frac{178801}{518400} \\ -\frac{65807}{518400} \\ \frac{25529}{64800} \\ -\frac{3521}{129600} \\ \frac{48769}{207360} \\ \frac{513}{12800} \end{pmatrix}$$

## Conflicts of Interest

The author(s) declare that there is no conflict of interest regarding the publication of this paper.

## Authors' Contributions

Conceptualization, S.J.; Methodology, S.J.; Software, A.E.O., A.V.O.; Validation, S.J., O.O.A. and A.E.O.; Formal Analysis, S.J.; Writing – Original Draft Preparation, S.J.; Writing – Review & Editing, S.J..; Visualization, S.J.; Supervision, S.J.;.

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## Appendix 1

$$\begin{pmatrix}\Lambda_{10}\\\Lambda_{11}\\\Lambda_{12}\\\Lambda_{13}\\\Lambda_{14}\\\Lambda_{15}\\\Lambda_{16}\\\Lambda_{17}\\\Lambda_{18}\\\Lambda_{19}\end{pmatrix}=\begin{pmatrix}-\frac{460423}{7302113280}\\\frac{781531}{3285950976\ 00}\\-\frac{14986661}{2738292480\ 0}\\\frac{194431}{829785600}\\-\frac{736139}{3651056640}\\\frac{7755449}{5476584960\ 0}\\-\frac{398609}{5476584960}\\\frac{231697}{9127641600}\\-\frac{586007}{1095316992\ 00}\\\frac{33769}{6571901952\ 0}\end{pmatrix},\begin{pmatrix}\Lambda_{20}\\\Lambda_{21}\\\Lambda_{22}\\\Lambda_{23}\\\Lambda_{24}\\\Lambda_{25}\\\Lambda_{26}\\\Lambda_{27}\\\Lambda_{28}\\\Lambda_{29}\end{pmatrix}=\begin{pmatrix}-\frac{11989}{7779240}\\\frac{4141}{256714920}\\-\frac{321631}{106964550}\\\frac{132563}{106964550}\\-\frac{209999}{213929100}\\\frac{27971}{42785820}\\-\frac{6931}{21392910}\\\frac{11771}{106964550}\\-\frac{9727}{427858200}\\\frac{2759}{1283574600}\end{pmatrix},\begin{pmatrix}\Lambda_{30}\\\Lambda_{31}\\\Lambda_{32}\\\Lambda_{33}\\\Lambda_{34}\\\Lambda_{35}\\\Lambda_{36}\\\Lambda_{37}\\\Lambda_{38}\\\Lambda_{39}\end{pmatrix}=\begin{pmatrix}-\frac{1348153}{1095316992\ 00}\\\frac{50471}{6571901952\ 0}\\-\frac{9326819}{2738292480\ 0}\\\frac{184957}{782369280}\\-\frac{7311389}{5476584960\ 0}\\\frac{911717}{1095316992\ 0}\\-\frac{1105133}{2738292480\ 0}\\\frac{33841}{2489356800}\\-\frac{61331}{2190633984\ 0}\\\frac{86909}{3285950976\ 00}\end{pmatrix},$$

$$\begin{pmatrix}\Lambda_{40}\\\Lambda_{41}\\\Lambda_{42}\\\Lambda_{43}\\\Lambda_{44}\\\Lambda_{45}\\\Lambda_{46}\\\Lambda_{47}\\\Lambda_{48}\\\Lambda_{49}\end{pmatrix}=\begin{pmatrix}-\frac{11321}{142619400}\\\frac{6451}{1283574600}\\-\frac{195709}{106964550}\\\frac{6199}{2376990}\\-\frac{1691}{2037420}\\\frac{117239}{213929100}\\-\frac{28603}{106964550}\\\frac{3217}{35654850}\\-\frac{1591}{85571640}\\\frac{41}{23337720}\end{pmatrix},\begin{pmatrix}\Lambda_{50}\\\Lambda_{51}\\\Lambda_{52}\\\Lambda_{53}\\\Lambda_{54}\\\Lambda_{55}\\\Lambda_{56}\\\Lambda_{57}\\\Lambda_{58}\\\Lambda_{59}\end{pmatrix}=\begin{pmatrix}-\frac{7587}{38635520}\\\frac{303}{24586240}\\-\frac{215757}{48294400}\\\frac{2881503}{338060800}\\-\frac{351189}{676121600}\\\frac{24723}{19317760}\\-\frac{41841}{67612160}\\\frac{71091}{338060800}\\-\frac{58887}{1352243200}\\\frac{5583}{1352243200}\end{pmatrix},\begin{pmatrix}\Lambda_{60}\\\Lambda_{61}\\\Lambda_{62}\\\Lambda_{63}\\\Lambda_{64}\\\Lambda_{65}\\\Lambda_{66}\\\Lambda_{67}\\\Lambda_{68}\\\Lambda_{69}\end{pmatrix}=\begin{pmatrix}-\frac{3928}{10696455}\\\frac{3692}{160446825}\\-\frac{40232}{4862025}\\\frac{959584}{53482275}\\-\frac{24392}{10696455}\\\frac{201136}{53482275}\\-\frac{248}{218295}\\\frac{21088}{53482275}\\-\frac{4364}{53482275}\\\frac{248}{32089365}\end{pmatrix},$$

$$\begin{pmatrix}\Lambda_{70}\\\Lambda_{71}\\\Lambda_{72}\\\Lambda_{73}\\\Lambda_{74}\\\Lambda_{75}\\\Lambda_{76}\\\Lambda_{77}\\\Lambda_{78}\\\Lambda_{79}\end{pmatrix}=\begin{pmatrix}-\frac{858875}{1460422656}\\\frac{482875}{1314380390\ 4}\\-\frac{14492875}{1095316992}\\\frac{11291125}{365105664}\\-\frac{494875}{66382848}\\\frac{20944625}{2190633984}\\-\frac{449125}{1095316992}\\\frac{234625}{365105664}\\-\frac{560375}{4381267968}\\\frac{159125}{1314380390\ 4}\end{pmatrix},\begin{pmatrix}\Lambda_{80}\\\Lambda_{81}\\\Lambda_{82}\\\Lambda_{83}\\\Lambda_{84}\\\Lambda_{85}\\\Lambda_{86}\\\Lambda_{87}\\\Lambda_{88}\\\Lambda_{89}\end{pmatrix}=\begin{pmatrix}-\frac{4563}{5282200}\\\frac{57}{1056440}\\-\frac{25569}{1320550}\\\frac{12519}{264110}\\-\frac{40041}{2641100}\\\frac{9747}{528220}\\-\frac{387}{120050}\\\frac{3051}{1320550}\\-\frac{171}{1056440}\\\frac{93}{5282200}\end{pmatrix},\begin{pmatrix}\Lambda_{90}\\\Lambda_{91}\\\Lambda_{92}\\\Lambda_{93}\\\Lambda_{94}\\\Lambda_{95}\\\Lambda_{96}\\\Lambda_{97}\\\Lambda_{98}\\\Lambda_{99}\end{pmatrix}=\begin{pmatrix}-\frac{54313}{45619200}\\\frac{10171}{136857600}\\-\frac{303899}{11404800}\\\frac{153811}{2280960}\\-\frac{115199}{4561920}\\\frac{699769}{22809600}\\-\frac{107947}{11404800}\\\frac{81011}{11404800}\\-\frac{959}{829440}\\\frac{1321}{27371520}\end{pmatrix},$$

$$\begin{pmatrix}X_{110}\\X_{111}\\X_{112}\\X_{113}\\X_{114}\\X_{115}\\X_{116}\\X_{117}\\X_{118}\\X_{119}\end{pmatrix}=\begin{pmatrix}-\frac{530113}{260789760}\\\frac{190073}{2933884800}\\\frac{10946503}{977961600}\\-\frac{5632757}{977961600}\\\frac{21995}{4346496}\\-\frac{7012483}{1955923200}\\\frac{120719}{65197440}\\-\frac{211151}{325987200}\\\frac{4181}{30561300}\\-\frac{30887}{2347107840}\end{pmatrix},\begin{pmatrix}X_{120}\\X_{121}\\X_{122}\\X_{123}\\X_{124}\\X_{125}\\X_{126}\\X_{127}\\X_{128}\\X_{129}\end{pmatrix}=\begin{pmatrix}\frac{72151}{3056130}\\\frac{8501}{7334712}\\\frac{129758}{7640325}\\\frac{4124}{7640325}\\\frac{112403}{30561300}\\-\frac{5899}{1528065}\\\frac{3676}{1528065}\\-\frac{7192}{7640325}\\\frac{1871}{8731800}\\-\frac{997}{45841950}\end{pmatrix},\begin{pmatrix}X_{130}\\X_{131}\\X_{132}\\X_{133}\\X_{134}\\X_{135}\\X_{136}\\X_{137}\\X_{138}\\X_{139}\end{pmatrix}=\begin{pmatrix}-\frac{1030627}{3911846400}\\\frac{19619}{1173553920}\\\frac{6050587}{977961600}\\\frac{171137}{27941760}\\-\frac{1513637}{488980800}\\\frac{148619}{78236928}\\-\frac{894727}{977961600}\\\frac{300263}{977961600}\\-\frac{24677}{391184640}\\\frac{69823}{1173553920\ 0}\end{pmatrix},$$

$$\begin{pmatrix} X_{140} \\ X_{141} \\ X_{142} \\ X_{143} \\ X_{144} \\ X_{145} \\ X_{146} \\ X_{147} \\ X_{148} \\ X_{149} \end{pmatrix} = \begin{pmatrix} -\frac{1648}{2546775} \\ \frac{7453}{183367800} \\ \frac{15791}{1091475} \\ \frac{43954}{1528065} \\ -\frac{251}{58212} \\ \frac{30158}{7640325} \\ -\frac{1699}{848925} \\ \frac{194}{282975} \\ \frac{1741}{12224520} \\ \frac{62}{4584195} \end{pmatrix}, \begin{pmatrix} X_{150} \\ X_{151} \\ X_{152} \\ X_{153} \\ X_{154} \\ X_{155} \\ X_{156} \\ X_{157} \\ X_{158} \\ X_{159} \end{pmatrix} = \begin{pmatrix} -\frac{9729}{9658880} \\ \frac{19}{301840} \\ \frac{272319}{12073600} \\ \frac{648267}{12073600} \\ \frac{128529}{12073600} \\ \frac{1143}{137984} \\ \frac{7923}{2414720} \\ \frac{13131}{12073600} \\ -\frac{2697}{12073600} \\ \frac{1019}{48294400} \end{pmatrix}, \begin{pmatrix} X_{160} \\ X_{161} \\ X_{162} \\ X_{163} \\ X_{164} \\ X_{165} \\ X_{166} \\ X_{167} \\ X_{168} \\ X_{169} \end{pmatrix} = \begin{pmatrix} -\frac{2092}{1528065} \\ \frac{1954}{22920975} \\ \frac{234296}{7640325} \\ \frac{599216}{7640325} \\ \frac{8524}{305613} \\ \frac{219992}{7640325} \\ -\frac{496}{218295} \\ \frac{9904}{7640325} \\ -\frac{2146}{7640325} \\ \frac{124}{4584195} \end{pmatrix}, \begin{pmatrix} X_{170} \\ X_{171} \\ X_{172} \\ X_{173} \\ X_{174} \\ X_{175} \\ X_{176} \\ X_{177} \\ X_{178} \\ X_{179} \end{pmatrix} = \begin{pmatrix} -\frac{3375}{1931776} \\ \frac{25625}{234710784} \\ \frac{1519675}{39118464} \\ \frac{4028875}{39118464} \\ \frac{147625}{3259872} \\ \frac{4004375}{78236928} \\ \frac{198875}{13039488} \\ \frac{46925}{13039488} \\ -\frac{5125}{11176704} \\ \frac{18875}{469421568} \end{pmatrix}, \begin{pmatrix} X_{180} \\ X_{181} \\ X_{182} \\ X_{183} \\ X_{184} \\ X_{185} \\ X_{186} \\ X_{187} \\ X_{188} \\ X_{189} \end{pmatrix} = \begin{pmatrix} -\frac{387}{188650} \\ \frac{19}{150920} \\ \frac{4404}{94325} \\ \frac{2424}{18865} \\ \frac{23013}{377300} \\ \frac{285}{3773} \\ \frac{3126}{94325} \\ \frac{2196}{94325} \\ \frac{171}{150920} \\ \frac{1}{188650} \end{pmatrix}, \begin{pmatrix} X_{190} \\ X_{191} \\ X_{192} \\ X_{193} \\ X_{194} \\ X_{195} \\ X_{196} \\ X_{187} \\ X_{198} \\ X_{199} \end{pmatrix} = \begin{pmatrix} -\frac{32431}{11404800} \\ \frac{1631}{8553600} \\ -\frac{161857}{2851200} \\ \frac{84329}{570240} \\ \frac{50029}{570240} \\ \frac{490931}{5702400} \\ \frac{179879}{2851200} \\ \frac{107093}{2851200} \\ \frac{6349}{285120} \\ \frac{1723}{1368576} \end{pmatrix},$$